Projection schemes in the null field method

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1 Introduction

Three-dimensional problems of electromagnetic scattering have been the subject of intense investigation and research. One of the fastest and most powerful numerical tools for computing the nonspherical light scattering is the null field method [1] (otherwise known as the extended boundary condition method, Schelkunoff equivalent current method, Eswald-Oseen extinction theorem and T-matrix method). A set of integral equations for the surface current densities is derived by considering the null-field condition for the scattered field inside the particle. The solution of the scattering problem is obtained by approximating the surface fields by the complete set of tangential single spherical coordinate vector wave functions. The theoretical foundations of the method were given by Ramm [2,3], and Kristensson, Ramm and Ström [4]. They include analysis of the convergence of the method, stability of the numerical scheme towards small perturbations of data and estimates of rate of convergence. Remarkably enough, particular satisfactory criteria for choosing the complete family to approximate the surface current densities were given. According to these criteria the single spherical coordinate vector wave functions can not be used for surface current approximation since in this case there are no guarantees that the scheme converge. However, it is noted that in practice, in computational simulations, the null field method converges for a widely class of particle shapes.

The subject matter of this paper is to construct convergent projection schemes in the framework of the null field method. These schemes are derived by applying the fundamental theorem of discrete approximation to different variational equations in $\mathcal{L}^2_{\tan}(S)$. Here, $\mathcal{L}^2_{\tan}(S)$ stands for the space of square integrable tangential vector functions on S. The formalism is valid for any system of functions which is complete on the particle surface. In this context , the method is general since in addition to the completeness no other requirements should be imposed. In the present paper we analyze the scattering by perfectly conducting particles, i.e. the exterior Maxwell boundary value problem.

2 Formulation

Let us consider the electromagnetic wave propagation in a homogeneous, isotropic medium that occupies the exterior of a bounded domain D_i in $\mathbf{R}^3$. The domain D_i has a closed boundary S, and a simply-connected exterior D_s. Suppose now that S is perfectly conducting and let $\mathbf{n}$ denote the outward unit normal to S. Let $\mathbf{E}_0, \mathbf{H}_0$ be an entire solution to the Maxwell equations representing an incident electromagnetic field.

The direct electromagnetic obstacle scattering problem can be formulated as follows: find a solution $\mathbf{E}_s$ and $\mathbf{H}_s$ to the Maxwell equation in D_s

$$\begin{aligned} \nabla \times \mathbf{E}_s &= jk_s\mathbf{H}_s \\ \nabla \times \mathbf{H}_s &= -jk_s\mathbf{E}_s, \end{aligned} \tag{1}$$

satisfying the Silver-Müller radiation condition

$$\frac{\mathbf{x}}{x} \times \mathbf{H}_s + \mathbf{E}_s = o\left(\frac{1}{x}\right), \quad \text{as } x \to \infty \tag{2}$$

uniformly for all directions $\mathbf{x}/x$,and the boundary conditions on S

$$\mathbf{n} \times \mathbf{E}_s + \mathbf{n} \times \mathbf{E}_0 = 0. \tag{3}$$

It is noted that the direct electromagnetic scattering problem is a particular case of the exterior Maxwell problem. If $Im\,k_s \geq 0$ and the boundary data belongs to $\mathcal{C}^{0,\alpha}_{\tan,d}(S)$, then there exists an unique solution to the exterior Maxwell boundary-value problem. The proof is given in [5,6]. Here, $\mathcal{C}^{0,\alpha}_{\tan,d}(S)$ is the space of all uniformly continuous tangential vector fields on S, $0 < \alpha \leq 1$ with Hölder-continuous surface divergence. For the scattering problem, the boundary values are the restriction of an analytic field $\mathbf{E}_0, \mathbf{H}_0$ to the boundary and therefore they are as smooth as the boundary. In our analysis we will assume that the surface S fulfilled sufficient smoothness requirements such that the solution $\mathbf{E}_s$, $\mathbf{H}_s \in C^{0,\alpha}(\overline{D}_s)$.

Let $\mathbf{E}_s, \mathbf{H}_s$ be the solution to the exterior Maxwell boundary-value problem. Application of Stratton-Chu representation theorem to $\mathbf{E}_s$ and

$\mathbf{H}_s$ gives

$$\begin{aligned} -\nabla\times\int_S \mathbf{e}_0(\mathbf{y})g(\mathbf{x},\mathbf{y},k_s)dS(\mathbf{y}) & \\ +\frac{j}{k_s}\nabla\times\nabla\times\int_S \mathbf{h}_s(\mathbf{y})g(\mathbf{x},\mathbf{y},k_s)dS(\mathbf{y}) &= 0 \end{aligned} \tag{4}$$

in D_i, where $\mathbf{e}_0 = \mathbf{n}\times\mathbf{E}_0$ and $\mathbf{E}_0, \mathbf{H}_0$ is an entire solution to the Maxwell equations representing an incident electromagnetic field. The above equation is called the null field equation for the surface density $\mathbf{h}_s$. The null field equation is based on the assumption that a solution to the boundary value problem exists ($\mathbf{h}_s = \mathbf{n}\times\mathbf{H}_s$), and hence the question of existence of solutions is obvious. It is also possible to prove the uniqueness, and the fact that, $\mathbf{h}_s \in \mathcal{C}^{0,\alpha}_{\tan,d}(S)$. For $\mathbf{h}_s \in \mathcal{C}^{0,\alpha}_{\tan,d}(S)$ solving (4) we construct the solution to the exterior Maxwell boundary-value problem by

$$\begin{aligned} \mathbf{E}_s(\mathbf{x}) &= -\nabla\times\int_S \mathbf{e}_0(\mathbf{y})g(\mathbf{x},\mathbf{y},k_s)dS(\mathbf{y}) \\ & +\frac{j}{k_s}\nabla\times\nabla\times\int_S \mathbf{h}_s(\mathbf{y})g(\mathbf{x},\mathbf{y},k_s)dS(\mathbf{y}) \end{aligned} \tag{5}$$

Note that $\mathbf{H}_s(\mathbf{x}) = (1/jk_s)\nabla\times\mathbf{E}_s(\mathbf{x})$, and $\mathbf{E}_s$ and $\mathbf{H}_s$ belong to $C^{0,\alpha}(\overline{D}_s)$. Let $\mathbf{h}_{sN}$ be an approximation of the surface field $\mathbf{h}_s$ and let $\mathbf{E}_s^N$ stands for an approximation of the scattered field $\mathbf{E}_s$. Then, obviously the following estimates hold

$$\left\|\mathbf{E}_s - \mathbf{E}_s^N\right\|_{\infty,G_s} \leq C\left\|\mathbf{h}_s - \mathbf{h}_{sN}\right\|_2 \tag{6}$$

for some constant C depending on S and any compact region $G_i \subset D_i$. Therefore, the approximate solution $\mathbf{E}_s^N$ converges to the exact solution $\mathbf{E}_s$, if $\mathbf{h}_{sN}$ converges in L^2-norm to $\mathbf{h}_s$.

Wriedt and Doicu [7] proved that the general null field equation (4) is equivalent to the following set of integral equations

$$\begin{aligned} \left\langle \mathbf{h}_s, \mathbf{n}\times\mathbf{n}\times\mathbf{\Psi}_\nu^{3*}\right\rangle_2 &= -j\left\langle \mathbf{e}, \mathbf{n}\times\mathbf{n}\times\mathbf{\Phi}_\nu^{3*}\right\rangle_2 \\ \left\langle \mathbf{h}_s, \mathbf{n}\times\mathbf{n}\times\mathbf{\Phi}_\nu^{3*}\right\rangle_2 &= -j\left\langle \mathbf{e}_0, \mathbf{n}\times\mathbf{n}\times\mathbf{\Psi}_\nu^{3*}\right\rangle_2. \end{aligned} \tag{7}$$

Here, $\langle\ ,\ \rangle_2$ denotes the scalar product in $\mathcal{L}^2_{\tan}(S)$, and the set $\{\mathbf{\Psi}_\nu^{1,3}, \mathbf{\Phi}_\nu^{1,3}\}$ stands for the localized vector spherical functions, distributed spherical vector wave functions, magnetic and electric dipoles,or vector Mie-potentials. By convention, when we refer to the null field equations (7) we refer implicitly to all equivalent forms of these equations.

Now, we are in position to discuss projection schemes for the general null field equations. At this end consider the null field equations (7) written for convenience as

$$\begin{aligned}\left\langle \mathbf{n}\times\mathbf{h}_s^*,\mathbf{n}\times\mathbf{\Psi}_\nu^3\right\rangle_2 &= j\left\langle \mathbf{n}\times\mathbf{e}_0^*,\mathbf{n}\times\mathbf{\Phi}_\nu^3\right\rangle_2 \\ \left\langle \mathbf{n}\times\mathbf{h}_s^*,\mathbf{n}\times\mathbf{\Phi}_\nu^3\right\rangle_2 &= j\left\langle \mathbf{n}\times\mathbf{e}_0^*,\mathbf{n}\times\mathbf{\Psi}_\nu^3\right\rangle_2\end{aligned} \tag{8}$$

for $\nu = 1, 2, \ldots$Let the sequence

$$\mathbf{h}'_{sN}=\sum_{\mu=1}^{N} a_\mu^N\left(\mathbf{n}\times\mathbf{\Psi}_\mu^3\right)+b_\mu^N\left(\mathbf{n}\times\mathbf{\Phi}_\mu^3\right) \tag{9}$$

solves the null field equations

$$\begin{aligned}\left\langle \mathbf{h}'_{sN},\mathbf{n}\times\mathbf{\Psi}_\nu^3\right\rangle_2 &= j\left\langle \mathbf{n}\times\mathbf{e}_0^*,\mathbf{n}\times\mathbf{\Phi}_\nu^3\right\rangle_2 \\ \left\langle \mathbf{h}'_{sN},\mathbf{n}\times\mathbf{\Phi}_\nu^3\right\rangle_2 &= j\left\langle \mathbf{n}\times\mathbf{e}_0^*,\mathbf{n}\times\mathbf{\Psi}_\nu^3\right\rangle_2\end{aligned} \tag{10}$$

for $\nu = 1, 2, , , , N$, and denote $\mathbf{h}'_s = \mathbf{n}\times\mathbf{h}_s^*$. Then, from (8) and (10) we see that $\mathbf{h}'_{sN}$ also satisfies the system of equations

$$\begin{aligned}\left\langle \mathbf{h}'_{sN}-\mathbf{h}'_s,\mathbf{n}\times\mathbf{\Psi}_\nu^3\right\rangle_2 &= 0 \\ \left\langle \mathbf{h}'_{sN}-\mathbf{h}'_s,\mathbf{n}\times\mathbf{\Phi}_\nu^3\right\rangle_2 &= 0\end{aligned} \tag{11}$$

$\nu = 1, 2, , , , N$, and therefore, according to theorem A.2.1 given in Appendix A, we conclude that $\|\mathbf{h}'_{sN}-\mathbf{h}'_s\|_2 \to 0$ as $N \to \infty$. Hence, the sequence $\mathbf{h}_{sN} = -\mathbf{n}\times\mathbf{h}'^*_{sN}$, converges strongly to $\mathbf{h}_s$. Similar arguments and application of theorem A.2.2 shows that the sequence $\mathbf{h}_{sN} = \mathbf{h}'^*_{sN}$, where

$$\begin{aligned}\mathbf{h}'_{sN} &= \sum_{\mu=1}^{N} a_\mu^N\left(\mathbf{n}\times\mathbf{\Psi}_\mu^3+\lambda\mathbf{n}\times\left(\mathbf{n}\times\mathbf{\Psi}_\mu^3\right)\right) \\ &\quad +b_\mu^N\left(\mathbf{n}\times\mathbf{\Phi}_\mu^3+\lambda\mathbf{n}\times\left(\mathbf{n}\times\mathbf{\Phi}_\mu^3\right)\right)\end{aligned} \tag{12}$$

solves the null field equations

$$\begin{aligned}\left\langle \mathbf{h}'_{sN},\mathbf{n}\times\left(\mathbf{n}\times\mathbf{\Psi}_\nu^3\right)\right\rangle_2 &= j\left\langle \mathbf{e}_0^*,\mathbf{n}\times\left(\mathbf{n}\times\mathbf{\Phi}_\nu^3\right)\right\rangle_2 \\ \left\langle \mathbf{h}'_{sN},\mathbf{n}\times\left(\mathbf{n}\times\mathbf{\Phi}_\nu^3\right)\right\rangle_2 &= j\left\langle \mathbf{e}_0^*,\mathbf{n}\times\left(\mathbf{n}\times\mathbf{\Psi}_\nu^3\right)\right\rangle_2\end{aligned} \tag{13}$$

$\nu = 1, 2, , , , N$, converges strongly to $\mathbf{h}_s$. In order to exploit the full content of theorem A.2 we see that according to part 3 the sequence

$$\begin{aligned}\mathbf{h}_{sN} &= \sum_{\mu=1}^{N} a_\mu^N\left(\mathbf{n}\times\mathbf{\Psi}_\mu^3\right)+\lambda a_\mu^{N*}\left(\mathbf{n}\times\left(\mathbf{n}\times\mathbf{\Psi}_\mu^{3*}\right)\right) \\ &\quad +b_\mu^N\left(\mathbf{n}\times\mathbf{\Phi}_\mu^3\right)+\lambda b_\mu^{N*}\left(\mathbf{n}\times\left(\mathbf{n}\times\mathbf{\Phi}_\mu^{3*}\right)\right)\end{aligned} \tag{14}$$

solving the null field equations

$$\begin{aligned}\left\langle \mathbf{h}_{sN}, \mathbf{n}\times\left(\mathbf{n}\times\mathbf{\Psi}_{\nu}^{3*}\right)\right\rangle_2 &= -j\left\langle \mathbf{e}_0, \mathbf{n}\times\left(\mathbf{n}\times\mathbf{\Phi}_{\nu}^{3*}\right)\right\rangle_2 \\ \left\langle \mathbf{h}_{sN}, \mathbf{n}\times\left(\mathbf{n}\times\mathbf{\Phi}_{\nu}^{3*}\right)\right\rangle_2 &= -j\left\langle \mathbf{e}_0^*, \mathbf{n}\times\left(\mathbf{n}\times\mathbf{\Psi}_{\nu}^{3*}\right)\right\rangle_2,\end{aligned} \tag{15}$$

converges in L^2-norm to $\mathbf{h}_s$.

We conclude this section by recalling the projection scheme of the conventional null field method. Let us write Eqs.(7) in terms of the total magnetic field $\mathbf{h} = \mathbf{h}_s + \mathbf{h}_0$ as

$$\begin{aligned}\left\langle \mathbf{h}-\mathbf{h}_0, \mathbf{n}\times\left(\mathbf{n}\times\mathbf{\Psi}_{\nu}^{3*}\right)\right\rangle_2 &= -j\left\langle \mathbf{e}_0, \mathbf{n}\times\left(\mathbf{n}\times\mathbf{\Phi}_{\nu}^{3*}\right)\right\rangle_2 \\ \left\langle \mathbf{h}-\mathbf{h}_0, \mathbf{n}\times\left(\mathbf{n}\times\mathbf{\Phi}_{\nu}^{3*}\right)\right\rangle_2 &= -j\left\langle \mathbf{e}_0, \mathbf{n}\times\left(\mathbf{n}\times\mathbf{\Psi}_{\nu}^{3*}\right)\right\rangle_2\end{aligned} \tag{16}$$

where $\nu = 1, 2, ...$and $\mathbf{h}_0 = \mathbf{n}\times\mathbf{H}_0$. Thus, by assuming that k_s is not an irregular frequency the approximate solution $\mathbf{h}_N$ can be sought in the form of a linear combination of regular fields

$$\mathbf{h}_N = \sum_{\mu=1}^{N} a_{\mu}^N(\mathbf{n}\times\mathbf{\Psi}_{\mu}^1) + b_{\mu}^N(\mathbf{n}\times\mathbf{\Phi}_{\mu}^1). \tag{17}$$

When localized vector spherical functions are used as basis and testing functions, the above projection method is identically to the scheme obtained in the frame of the single spherical coordinate-based null field method. Therefore, by convention, the projection method (16) with localized vector spherical functions, distributed spherical vector wave functions, magnetic and electric dipoles and vector Mie-potentials will be referred to as the conventional null field method with discrete sources.

Projection schemes (9)-(10), (12)-(13) and (14)- (15) will be referred to as PS1, PS2 and PS3, respectively. Obviously, projection scheme PS1 corresponds to the least square method. The same happen with projection schemes PS2 and PS3 for $\lambda \to \infty$. In addition, for $\lambda = 0$ PS2 and PS3 are similar in the sense that $\mathbf{h}_{sN}$ satisfies (15) but the explicit form of $\mathbf{h}_{sN}$ is

$$\mathbf{h}_{sN} = \sum_{\mu=1}^{N} a_{\mu}^N\left(\mathbf{n}\times\mathbf{\Psi}_{\mu}^{3*}\right) + b_{\mu}^N\left(\mathbf{n}\times\mathbf{\Phi}_{\mu}^{3*}\right) \tag{18}$$

for projection scheme PS2, and

$$\mathbf{h}_{sN} = \sum_{\mu=1}^{N} a_{\mu}^N\left(\mathbf{n}\times\mathbf{\Psi}_{\mu}^{3}\right) + b_{\mu}^N\left(\mathbf{n}\times\mathbf{\Phi}_{\mu}^{3}\right) \tag{19}$$

for projection scheme PS3. It is noted that if $\lambda = 0$ our convergence proof fails since the associated sesquilinear form is not coercive. In this case the convergence can be proved by assuming the validity of the Rayleigh hypothesis.

Numerical simulations shown that the projection scheme corresponding to the least square method has a low rate of convergence. The convergence can be accelerated by using a system of vector functions consisting in a linear combination of tangential vector fields and their rotated components about the unit normal. The parameter λ appearing in these schemes can be used for controling the convergence rate and therefore gives more flexibility to these algorithms. In addition, the numerical analysis shown that the conventional null field method is the most efficient in spite of the fact that we are not able to prove the convergence. In this context the convergence question of the conventional method is open and several addition research is needed. Our contribution demonstrate once again the geniale feeling of Waterman who chose simple derivation, preferring physical plausibility over mathematical rigor. Those who demand the latter are reminded that 'one man's rigor is another man's mortis'.

3 Appendix A

In this appendix we present some fundamental results from linear functional analysis.

Let H be a Hilbert space. The mapping $\mathcal{B} : H \times H \to C$ is called a sequilinear form on H if it is linear in the first argument and antilinear or semilinear in the second one, i.e.

$$\begin{aligned} \mathcal{B}(\alpha x_1 + \beta x_2, y) \quad & \alpha \mathcal{B}(x_1, y) + \beta \mathcal{B}(x_2, y) \\ \mathcal{B}(x, \alpha y_1 + \beta y_2) \quad & \alpha^* \mathcal{B}(x, y_1) + \beta^* \mathcal{B}(x, y_2) \end{aligned} \tag{20}$$

A sesquilinear form $\mathcal{B}$ is bounded, if it exists a real constant $M > 0$ such that

$$|\mathcal{B}(x, y)| \leq M \, \|x\|_H \, \|y\|_H \tag{21}$$

and coercive or H-elliptic, if it exists a real constant $c > 0$ such that

$$\operatorname{Re} \mathcal{B}(x, x) \geq c \, \|x\|_H^2 \tag{22}$$

Theorem A.1(Fundamental Theorem of Discrete Approximation) Let H be a Hilbert space, $\mathcal{B}$ a bounded and coercive sesquilinear form on H, $\mathcal{F}$ a linear and continuous functional on H, and $\{\psi_i\}_{i=1}^{\infty}$ a complete and linearly independent system in H. Then

1. the algebraic system of equations

$$\sum_{i=1}^{N} \mathcal{B}(\psi_i, \psi_j) \, a_i^N = \mathcal{F}^*(\psi_j) \, , \;\; j = 1, .., N \tag{23}$$

has an unique solution;

2. the sequence $u_N = \sum_{i=1}^{N} \alpha_i^N \psi_i$ is convergent, and if $\|u_N - u\|_H \to 0$ as $N \to \infty$, then

$$\mathcal{F}^*(x) = \mathcal{B}(u, x), \ \forall x \in H \tag{24}$$

The proof of the above theorem can be found in [8].

Let us apply the fundamental theorem of discrete approximation to the Hilbert space $\mathcal{L}^2_{\tan}(S)$. Our aim is to construct convergent projection methods for the variational problems

$$\begin{aligned}
&1. \quad \langle \mathbf{u} - \mathbf{u}_0, \mathbf{x} \rangle_2 = 0, \forall \mathbf{x} \in \mathcal{L}^2_{\tan}(S) \\
&2. \quad \langle \mathbf{u} - \mathbf{u}_0, \mathbf{n} \times \mathbf{x} + \lambda \mathbf{x} \rangle_2 = 0, \forall \mathbf{x} \in \mathcal{L}^2_{\tan}(S), \ \lambda > 0 \\
&3. \quad \langle (\mathbf{u} - \mathbf{u}_0) + \lambda \mathbf{n} \times (\mathbf{u}^* - \mathbf{u}_0^*), \mathbf{n} \times \mathbf{x}^* \rangle_2 = 0, \forall \mathbf{x} \in \mathcal{L}^2_{\tan}(S), \ \lambda > 0
\end{aligned} \tag{25}$$

Let us assume that the surface S is of class C^2. If S has the parametric representation $\mathbf{x} = \mathbf{x}(u, v)$ at each point on the surface we can define an orthogonal tangent-normal system of unit vectors $(\mathbf{e}_u, \mathbf{e}_v, \mathbf{n})$, where $\mathbf{n}$ represents the outward unit normal vector to S, and $\mathbf{e}_u$ and $\mathbf{e}_v$ are orthogonal unit vectors in the tangent plane of S.

Projection methods in $\mathcal{L}^2_{\tan}(S)$ are given by the following theorem

Theorem A.2 Let $\{\mathbf{\Psi}_i\}_{i=1}^{\infty}$ be a complete system of vector functions in $\mathcal{L}^2_{\tan}(S)$ and $\mathbf{u}_0 \in \mathcal{L}^2_{\tan}(S)$. The sequence $\mathbf{u}_N = \sum_{i=1}^{N} \alpha_i^N \mathbf{\Psi}_i$, satisfying the projection relations

$$\begin{aligned}
&1. \quad \langle \mathbf{u}_N - \mathbf{u}_0, \mathbf{\Psi}_j \rangle_2 = 0, 1 \le j \le N \\
&2. \quad \langle \mathbf{u}_N - \mathbf{u}_0, \mathbf{n} \times \mathbf{\Psi}_j + \lambda \mathbf{\Psi}_j \rangle_2 = 0, 1 \le j \le N \\
&3. \quad \left\langle (\mathbf{u}_N - \mathbf{u}_0) + \lambda \mathbf{n} \times (\mathbf{u}_N^* - \mathbf{u}_0^*), \mathbf{n} \times \mathbf{\Psi}_j^* \right\rangle_2 = 0, 1 \le j \le N
\end{aligned} \tag{26}$$

converge in the L_2−norm to $\mathbf{u}_0$, i.e.

$$\|\mathbf{u}_N - \mathbf{u}_0\|_2 \to 0 \text{ as } N \to \infty \tag{27}$$

4 References

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