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Projection schemes in the null field method

A. Doicu[a,*], Yu. A. Eremin[b], T. Wriedt[a]

[a]*Institut für Werkstofftechnik, Badgasteiner Straße 3, 28359 Bremen, Germany*
[b]*Department of Applied Mathematics and Computer Science, Moscow State Univerity, Vorobyov Hills, 119899 Moscow, Russia*

Abstract

Projection schemes for constructing an approximate solution to the exterior Maxwell boundary problem are presented. The methods are derived in the context of the null field approach by using the fundamental theorem of discrete approximation. We established the convergence and unique solvability of the linear system of equations for any system of functions which is complete on the particle surface. The convergence of the projection schemes is analyzed from a computational point of view.

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1. Introduction

Three-dimensional problems of electromagnetic scattering have been the subject of intense investigation and research. One of the fastest and most powerful numerical tools for computing the nonspherical light scattering is the null field method [1] (otherwise known as the extended boundary condition method, Schelkunoff equivalent current method, Eswald–Oseen extinction theorem and T-matrix method). A set of integral equations for the surface current densities is derived by considering the null-field condition for the scattered field inside the particle. The solution of the scattering problem is obtained by approximating the surface fields by the complete set of tangential single spherical coordinate vector wave functions. The theoretical foundations of the method were given by Ramm [2,3], and Kristensson et al. [4]. They include analysis of the convergence of the method, stability of the numerical scheme towards small perturbations of data and estimates of rate of convergence. Remarkably enough, particular satisfactory criteria for choosing the complete family to approximate the surface current densities were given. According to

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* Corresponding author. Tel.: + 49-421-218-4890; fax: + 49-421-218-5378
E-mail address: doicu@iwt.uni-bremen.de (A. Doicu)

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these criteria the single spherical coordinate vector wave functions cannot be used for surface current approximation since in this case there are no guarantees that the scheme converge. However, it is noted that in practice, in computational simulations, the null field method converges for a wide class of particle shapes.

The subject matter of this paper is to construct convergent projection schemes in the framework of the null field method. These schemes are derived by applying the fundamental theorem of discrete approximation to different variational equations in $\mathscr{L}^2_{\tan}(S)$. Here, $\mathscr{L}^2_{\tan}(S)$ stands for the space of square integrable tangential vector functions on S. The formalism is valid for any system of functions which is complete on the particle surface. In this context, the method is general since in addition to the completeness no other requirements should be imposed. In the present paper we analyze the scattering by perfectly conducting particles, i.e. the exterior Maxwell boundary-value problem.

2. Mathematical formulation

Let us consider the electromagnetic wave propagation in a homogeneous, isotropic medium that occupies the exterior of a bounded domain D_{i} in $\mathbf{R}^3$. The domain D_{i} has a closed boundary S, and a simply-connected exterior D_{s}. Suppose now that S is perfectly conducting and let $\mathbf{n}$ denote the outward unit normal to S. Let $\mathbf{E}_0, \mathbf{H}_0$ be an entire solution to the Maxwell equations representing an incident electromagnetic field.

The direct electromagnetic obstacle scattering problem can be formulated as follows: find a solution $\mathbf{E}_{\mathrm{s}}$ and $\mathbf{H}_{\mathrm{s}}$ to the Maxwell equation in D_{s}

$$\begin{aligned} \nabla \times \mathbf{E}_{\mathrm{s}} &= jk_{\mathrm{s}}\mathbf{H}_{\mathrm{s}}, \\ \nabla \times \mathbf{H}_{\mathrm{s}} &= -jk_{\mathrm{s}}\mathbf{E}_{\mathrm{s}}, \end{aligned} \tag{1}$$

satisfying the Silver–Müller radiation condition

$$\frac{\mathbf{x}}{x} \times \mathbf{H}_{\mathrm{s}} + \mathbf{E}_{\mathrm{s}} = o\left(\frac{1}{x}\right), \quad \text{as } x \to \infty \tag{2}$$

uniformly for all directions $\mathbf{x}/x$, and the boundary conditions on S

$$\mathbf{n} \times \mathbf{E}_{\mathrm{s}} + \mathbf{n} \times \mathbf{E}_0 = 0. \tag{3}$$

It is noted that the direct electromagnetic scattering problem is a particular case of the exterior Maxwell problem. If $\operatorname{Im} k_{\mathrm{s}} \geq 0$ and the boundary data belongs to $\mathscr{C}^{0,\alpha}_{\tan,\mathrm{d}}(S)$, then there exists an unique solution to the exterior Maxwell boundary-value problem. The proof is given in [5,6]. Here, $\mathscr{C}^{0,\alpha}_{\tan,\mathrm{d}}(S)$ is the space of all uniformly Hölder continuous tangential vector fields on S, $0 < \alpha \leq 1$ with Hölder-continuous surface divergence. For the scattering problem, the boundary values are the restriction of an analytic field $\mathbf{E}_0, \mathbf{H}_0$ to the boundary and therefore, they are as smooth as the boundary. In our analysis we will assume that the surface S fulfilled sufficient smoothness requirements such that the solution $\mathbf{E}_{\mathrm{s}}$, $\mathbf{H}_{\mathrm{s}} \in C^{0,\alpha}(\bar{D}_{\mathrm{s}})$.

Let $\mathbf{E}_s, \mathbf{H}_s$ be the solution to the exterior Maxwell boundary-value problem. Application of Stratton–Chu representation theorem to $\mathbf{E}_s$ and $\mathbf{H}_s$ gives for $\mathbf{x} \in D_i$

$$\nabla \times \int_S [\mathbf{n}(\mathbf{y}) \times \mathbf{E}_s(\mathbf{y})] g(\mathbf{x}, \mathbf{y}, k_s) \, dS(\mathbf{y}) + \frac{j}{k_s} \nabla \times \nabla \times \int_S [\mathbf{n}(\mathbf{y}) \times \mathbf{H}_s(\mathbf{y})] g(\mathbf{x}, \mathbf{y}, k_s) \, dS(\mathbf{y}) = 0. \tag{4}$$

In view of (4) we can state the following formulation of the so-called general null field equation: find a tangential field $\mathbf{h}_s$ satisfying the integral equation,

$$-\nabla \times \int_S \mathbf{e}_0(\mathbf{y}) g(\mathbf{x}, \mathbf{y}, k_s) \, dS(\mathbf{y}) + \frac{j}{k_s} \nabla \times \nabla \times \int_S \mathbf{h}_s(\mathbf{y}) g(\mathbf{x}, \mathbf{y}, k_s) \, dS(\mathbf{y}) = 0 \tag{5}$$

in D_i, where $\mathbf{e}_0 = \mathbf{n} \times \mathbf{E}_0$ and $\mathbf{E}_0, \mathbf{H}_0$ is an entire solution to the Maxwell equations representing an incident electromagnetic field. The null field equation is based on the assumption that a solution to the boundary value problem exists ($\mathbf{h}_s = \mathbf{n} \times \mathbf{H}_s$), and hence the question of existence of solutions is obvious. It is also possible to prove the uniqueness, and the fact that, $\mathbf{h}_s \in \mathscr{C}^{0,\alpha}_{\mathrm{tan,d}}(S)$.

For $\mathbf{h}_s \in \mathscr{C}^{0,\alpha}_{\mathrm{tan,d}}(S)$ solving (5) we construct the solution to the exterior Maxwell boundary-value problem by

$$\mathbf{E}_s(\mathbf{x}) = -\nabla \times \int_S \mathbf{e}_0(\mathbf{y}) g(\mathbf{x}, \mathbf{y}, k_s) \, dS(\mathbf{y}) + \frac{j}{k_s} \nabla \times \nabla \times \int_S \mathbf{h}_s(\mathbf{y}) g(\mathbf{x}, \mathbf{y}, k_s) \, dS(\mathbf{y}). \tag{6}$$

Note that $\mathbf{H}_s(\mathbf{x}) = (1/jk_s) \nabla \times \mathbf{E}_s(\mathbf{x})$, and $\mathbf{E}_s$ and $\mathbf{H}_s$ belong to $C^{0,\alpha}(\bar{D}_s)$. Let $\mathbf{h}_{sN}$ be an approximation of the surface field $\mathbf{h}_s$. Define the approximate and the residual electric field in D_s, by

$$\mathbf{E}_s^N(\mathbf{x}) = -\nabla \times \int_S \mathbf{e}_0(\mathbf{y}) g(\mathbf{x}, \mathbf{y}, k_s) \, dS(\mathbf{y}) + \frac{j}{k_s} \nabla \times \nabla \times \int_S \mathbf{h}_{sN}(\mathbf{y}) g(\mathbf{x}, \mathbf{y}, k_s) \, dS(\mathbf{y}) \tag{7}$$

and

$$\delta \mathbf{E}_s^N(\mathbf{x}) = -\nabla \times \int_S \mathbf{e}_0(\mathbf{y}) g(\mathbf{x}, \mathbf{y}, k_s) \, dS(\mathbf{y}) + \frac{j}{k_s} \nabla \times \nabla \times \int_S \mathbf{h}_{sN}(\mathbf{y}) g(\mathbf{x}, \mathbf{y}, k_s) \, dS(\mathbf{y}), \tag{8}$$

respectively. Then, obviously the following estimates hold

$$\|\mathbf{E}_s - \mathbf{E}_s^N\|_{\infty, G_s} \leq C \|\mathbf{h}_s - \mathbf{h}_{sN}\|_2, \qquad \|\delta \mathbf{E}_s^N\|_{\infty, G_i} \leq C' \|\mathbf{h}_s - \mathbf{h}_{sN}\|_2, \tag{9}$$

for some constants C and C' depending on S and any compact regions $G_s \subset D_s$ and $G_i \subset D_i$. Therefore, the approximate solution $\mathbf{E}_s^N$ converges to the exact solution $\mathbf{E}_s$, if $\mathbf{h}_{sN}$ converges in L^2-norm to $\mathbf{h}_s$.

Wriedt and Doicu [7] proved that the general null-field equation (5) is equivalent to the following set of integral equations

$$\begin{aligned} \langle \mathbf{h}_s, \mathbf{n} \times \mathbf{n} \times \mathbf{\Psi}_\nu^{3*} \rangle_2 &= -j \langle \mathbf{e}_0, \mathbf{n} \times \mathbf{n} \times \mathbf{\Phi}_\nu^{3*} \rangle_2, \\ \langle \mathbf{h}_s, \mathbf{n} \times \mathbf{n} \times \mathbf{\Phi}_\nu^{3*} \rangle_2 &= -j \langle \mathbf{e}_0, \mathbf{n} \times \mathbf{n} \times \mathbf{\Psi}_\nu^{3*} \rangle_2. \end{aligned} \tag{10}$$

Here, $\mathbf{n}\times\mathbf{n}\times\mathbf{a} = \mathbf{n}\times(\mathbf{n}\times\mathbf{a})$, $\langle\,,\rangle_2$ denotes the scalar product in $\mathscr{L}^2_{\tan}(S)$, and the set $\{\mathbf{\Psi}_\nu^{1,3}, \mathbf{\Phi}_\nu^{1,3}\}_{\nu=1,2,\ldots}$ stands for the localized vector spherical functions $\{\mathbf{M}_{mn}^{1,3}, \mathbf{N}_{mn}^{1,3}\}_{m\in\mathbf{Z}, n\geq\max(1,|m|)}$, distributed spherical vector-wave functions $\{\mathscr{M}_{mn}^{3,1}, \mathscr{N}_{mn}^{3,1}\}_{m\in\mathbf{Z}, n=1,2,\ldots}$, magnetic and electric dipoles $\{\mathscr{M}_{ni}^{1,3}, \mathscr{N}_{ni}^{1,3}\}_{n=1,2,\ldots,i=1,2}$, or vector Mie-potentials $\{\mathscr{M}_{n}^{1,3}, \mathscr{N}_{n}^{1,3}\}_{n=1,2,\ldots}$. By convention, when we refer to the null-field equations (10) we refer implicitly to all equivalent forms of these equations.

Now, we are in position to discuss projection schemes for the general null field equations. At this end consider the null-field equations (10) written for convenience as

$$\begin{aligned}\langle \mathbf{n}\times\mathbf{h}_s^*, \mathbf{n}\times\mathbf{\Psi}_\nu^3\rangle_2 &= j\langle \mathbf{n}\times\mathbf{e}_0^*, \mathbf{n}\times\mathbf{\Phi}_\nu^3\rangle_2,\\ \langle \mathbf{n}\times\mathbf{h}_s^*, \mathbf{n}\times\mathbf{\Phi}_\nu^3\rangle_2 &= j\langle \mathbf{n}\times\mathbf{e}_0^*, \mathbf{n}\times\mathbf{\Psi}_\nu^3\rangle_2,\end{aligned} \tag{11}$$

for $\nu = 1, 2, \ldots$. Let the sequence

$$\mathbf{h}'_{sN} = \sum_{\mu=1}^{N} a_\mu^N(\mathbf{n}\times\mathbf{\Psi}_\mu^3) + b_\mu^N(\mathbf{n}\times\mathbf{\Phi}_\mu^3) \tag{12}$$

solves the null-field equations

$$\begin{aligned}\langle \mathbf{h}'_{sN}, \mathbf{n}\times\mathbf{\Psi}_\nu^3\rangle_2 &= j\langle \mathbf{n}\times\mathbf{e}_0^*, \mathbf{n}\times\mathbf{\Phi}_\nu^3\rangle_2,\\ \langle \mathbf{h}'_{sN}, \mathbf{n}\times\mathbf{\Phi}_\nu^3\rangle_2 &= j\langle \mathbf{n}\times\mathbf{e}_0^*, \mathbf{n}\times\mathbf{\Psi}_\nu^3\rangle_2,\end{aligned} \tag{13}$$

for $\nu = 1, 2, \ldots, N$, and denote $\mathbf{h}'_s = \mathbf{n}\times\mathbf{h}_s^*$. Then, from (11) and (13) we see that $\mathbf{h}'_{sN}$ also satisfies the system of equations

$$\begin{aligned}\langle \mathbf{h}'_{sN} - \mathbf{h}'_s, \mathbf{n}\times\mathbf{\Psi}_\nu^3\rangle_2 &= 0,\\ \langle \mathbf{h}'_{sN} - \mathbf{h}'_s, \mathbf{n}\times\mathbf{\Phi}_\nu^3\rangle_2 &= 0,\end{aligned} \tag{14}$$

$\nu = 1, 2, \ldots, N$, and therefore, according to Theorem A.2.1 given in Appendix A, we conclude that $\|\mathbf{h}'_{sN} - \mathbf{h}'_s\|_2 \to 0$ as $N \to \infty$. Hence, the sequence $\mathbf{h}_{sN} = -\mathbf{n}\times\mathbf{h}'_{sN}{}^*$, converges strongly to $\mathbf{h}_s$.

Similar arguments and application of Theorem A.2.2 shows that the sequence $\mathbf{h}_{sN} = \mathbf{h}'_{sN}{}^*$, where

$$\mathbf{h}'_{sN} = \sum_{\mu=1}^{N} a_\mu^N(\mathbf{n}\times\mathbf{\Psi}_\mu^3 + \lambda\mathbf{n}\times\mathbf{n}\times\mathbf{\Psi}_\mu^3) + b_\mu^N(\mathbf{n}\times\mathbf{\Phi}_\mu^3 + \lambda\mathbf{n}\times\mathbf{n}\times\mathbf{\Phi}_\mu^3) \tag{15}$$

solves the null-field equations

$$\begin{aligned}\langle \mathbf{h}'_{sN}, \mathbf{n}\times\mathbf{n}\times\mathbf{\Psi}_\nu^3\rangle_2 &= j\langle \mathbf{e}_0^*, \mathbf{n}\times\mathbf{n}\times\mathbf{\Phi}_\nu^3\rangle_2,\\ \langle \mathbf{h}'_{sN}, \mathbf{n}\times\mathbf{n}\times\mathbf{\Phi}_\nu^3\rangle_2 &= j\langle \mathbf{e}_0^*, \mathbf{n}\times\mathbf{n}\times\mathbf{\Psi}_\nu^3\rangle_2,\end{aligned} \tag{16}$$

$\nu = 1, 2, \ldots, N$, converges strongly to $\mathbf{h}_s$.

In order to exploit the full content of Theorem A.2 we see that according to part 3 the sequence

$$\mathbf{h}_{sN} = \sum_{\mu=1}^{N} a_\mu^N(\mathbf{n}\times\mathbf{\Psi}_\mu^3) + \lambda a_\mu^{N*}(\mathbf{n}\times\mathbf{n}\times\mathbf{\Psi}_\mu^{3*}) + b_\mu^N(\mathbf{n}\times\mathbf{\Phi}_\mu^3) + \lambda b_\mu^{N*}(\mathbf{n}\times\mathbf{n}\times\mathbf{\Phi}_\mu^{3*}) \tag{17}$$

solving the null-field equations

$$\begin{aligned}\langle \mathbf{h}_{sN}, \mathbf{n}\times\mathbf{n}\times\mathbf{\Psi}_\nu^{3*}\rangle_2 &= -j\langle \mathbf{e}_0, \mathbf{n}\times\mathbf{n}\times\mathbf{\Phi}_\nu^{3*}\rangle_2,\\ \langle \mathbf{h}_{sN}, \mathbf{n}\times\mathbf{n}\times\mathbf{\Phi}_\nu^{3*}\rangle_2 &= -j\langle \mathbf{e}_0^*, \mathbf{n}\times\mathbf{n}\times\mathbf{\Psi}_\nu^{3*}\rangle_2,\end{aligned} \tag{18}$$

converges in L^2-norm to $\mathbf{h}_s$.

We conclude this section by recalling the projection scheme of the conventional null-field method. Let us write Eqs. (10) in terms of the total magnetic field $\mathbf{h} = \mathbf{h}_s + \mathbf{h}_0$ as

$$\begin{aligned}\langle \mathbf{h}-\mathbf{h}_0, \mathbf{n}\times\mathbf{n}\times\mathbf{\Psi}_\nu^{3*}\rangle_2 &= -j\langle \mathbf{e}_0, \mathbf{n}\times\mathbf{n}\times\mathbf{\Phi}_\nu^{3*}\rangle_2,\\ \langle \mathbf{h}-\mathbf{h}_0, \mathbf{n}\times\mathbf{n}\times\mathbf{\Phi}_\nu^{3*}\rangle_2 &= -j\langle \mathbf{e}_0, \mathbf{n}\times\mathbf{n}\times\mathbf{\Psi}_\nu^{3*}\rangle_2,\end{aligned} \tag{19}$$

where $\nu = 1, 2, \ldots$ and $\mathbf{h}_0 = \mathbf{n}\times\mathbf{H}_0$. Thus, by assuming that k_s is not an irregular frequency the approximate solution $\mathbf{h}_N$ can be sought in the form of a linear combination of regular fields

$$\mathbf{h}_N = \sum_{\mu=1}^{N} a_\mu^N(\mathbf{n}\times\mathbf{\Psi}_\mu^1) + b_\mu^N(\mathbf{n}\times\mathbf{\Phi}_\mu^1). \tag{20}$$

When localized vector spherical functions are used as basis and testing functions, the above projection method is identical to the scheme obtained in the frame of the single spherical coordinate-based null-field method. Therefore, by convention, the projection method (19) with localized vector spherical functions, distributed spherical vector wave functions, magnetic and electric dipoles and vector Mie-potentials will be referred to as the conventional null field method with discrete sources. The resulting projection method is not on the same type as those derived on basis of the discrete approximation theorem, because the testing functions may differ from the basis functions. The convergence of the above projection methods can be analyzed in the same manner as Ramm did in the acoustic case [2,3]. The conclusion of this analysis is that the projection method (19)–(20) converges if the systems $\{\mathbf{n}\times\mathbf{\Psi}_\nu^3, \mathbf{n}\times\mathbf{\Phi}_\nu^3\}_{\nu=1}^\infty$ and $\{\mathbf{n}\times\mathbf{\Psi}_\nu^1, \mathbf{n}\times\boldsymbol{\Phi}_\nu^1\}_{\nu=1}^\infty$ form a Riesz basis of $\mathscr{L}_{\tan}^2(S)$ and the spectral condition

$$\inf_N \lambda((\mathbf{A}'\mathbf{A})_N) \geq \lambda > 0 \tag{21}$$

holds. Here $(\mathbf{A})_N$ is the truncated matrix

$$(\mathbf{A})_N = \begin{bmatrix} A_{\nu\mu}^{11} & A_{\nu\mu}^{12} \\ A_{\nu\mu}^{21} & A_{\nu\mu}^{22} \end{bmatrix}, \quad 1 \leq \nu,\ \mu \leq N \tag{22}$$

with

$$\begin{aligned} A_{\nu\mu}^{11} &= \langle \mathbf{n}\times\mathbf{\Psi}_\mu^1, \mathbf{n}\times\mathbf{n}\times\mathbf{\Psi}_\nu^{3*}\rangle_2, \quad & A_{\nu\mu}^{12} &= \langle \mathbf{n}\times\mathbf{\Phi}_\mu^1, \mathbf{n}\times\mathbf{n}\times\mathbf{\Psi}_\nu^{3*}\rangle_2,\\ A_{\nu\mu}^{21} &= \langle \mathbf{n}\times\mathbf{\Psi}_\mu^1, \mathbf{n}\times\mathbf{n}\times\mathbf{\Phi}_\nu^{3*}\rangle_2, \quad & A_{\nu\mu}^{22} &= \langle \mathbf{n}\times\mathbf{\Phi}_\mu^1, \mathbf{n}\times\mathbf{n}\times\mathbf{\Phi}_\nu^{3*}\rangle_2,\end{aligned} \tag{23}$$

and $(\mathbf{A}'\mathbf{A})_N = (\mathbf{A}')_N(\mathbf{A})_N$, where $\mathbf{A}'$ is the conjugate transpose of the matrix $\mathbf{A}$, i.e. $A'_{\mu\nu} = A^*_{\nu\mu}$. The systems of radiating and regular vector spherical functions do not form either a Riesz basis or a Schauder basis. A counterexample showing that the spherical vector wave functions do not form a basis in $\mathscr{L}^2_{\tan}(S)$ is given in Appendix B. Therefore, in view of Ramm's analysis we cannot say anything about the convergence of conventional null-field method.

Projection schemes (12)–(13),(15)–(16) and (17)–(18) will be referred to as PS1, PS2 and PS3, respectively. Obviously, projection scheme PS1 corresponds to the least-squares method. The same happened with projection schemes PS2 and PS3 for $\lambda \to \infty$. In addition, for $\lambda = 0$ PS2 and PS3 are similar in the sense that $\mathbf{h}_{sN}$ satisfies (18) but the explicit form of $\mathbf{h}_{sN}$ is

$$\mathbf{h}_{sN} = \sum_{\mu=1}^{N} a_\mu^N(\mathbf{n} \times \mathbf{\Psi}_\mu^{3*}) + b_\mu^N(\mathbf{n} \times \mathbf{\Phi}_\mu^{3*}), \tag{24}$$

for projection scheme PS2, and

$$\mathbf{h}_{sN} = \sum_{\mu=1}^{N} a_\mu^N(\mathbf{n} \times \mathbf{\Psi}_\mu^{3}) + b_\mu^N(\mathbf{n} \times \mathbf{\Phi}_\mu^{3}), \tag{25}$$

for projection scheme PS3. It is noted that if $\lambda = 0$ our convergence proof fails since the associated sesquilinear form is not coercive. In this case the convergence can be proved by assuming the validity of the Rayleigh hypothesis.

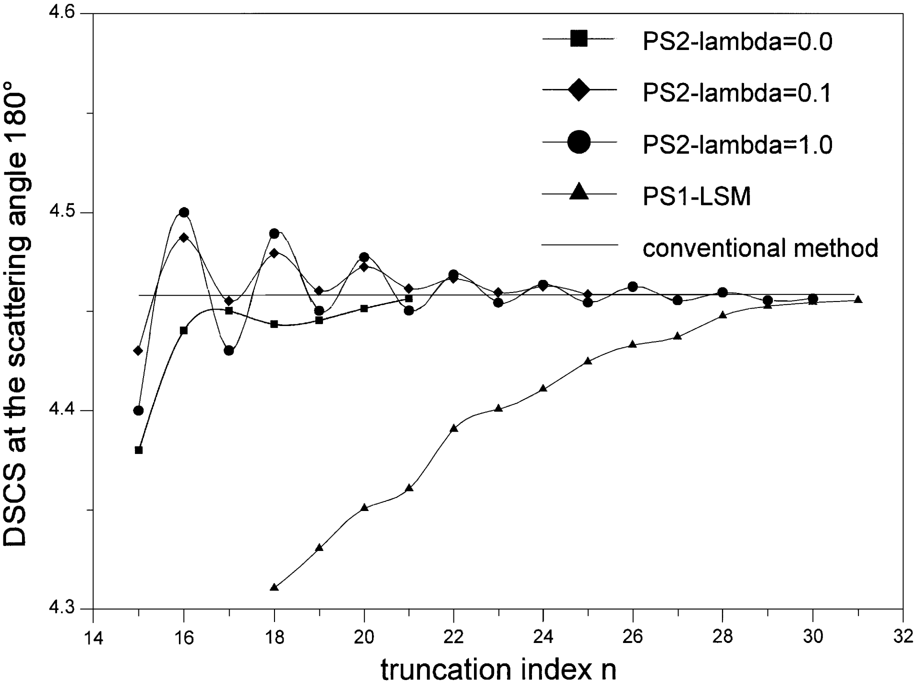

Fig. 1. Normalized differential scattering cross section (DSCS) at the scattering angle $\theta = 180°$ for different values of the truncation index. The scatterer is a prolate spheroid with semiaxes $k_s a = 5$ and $k_s b = 3$. The curves are computed with the conventional null-field method and projection scheme PS2. Localized spherical vector wave functions are used for representing the solution.

3. Numerical simulations

In this section we present some computer simulations in order to give a clear picture on the convergence of the above projection schemes.

In our first example we consider a prolate spheroid with semiaxes $k_s a = 5$ and $k_s b = 3$. The incident field is a plane wave propagating along the particle symmetry axis. The surface current density is approximated by linear combinations of localized spherical vector wave functions $\mathbf{M}_{mn}^{1,3}$ and $\mathbf{N}_{mn}^{1,3}$, with $m = -m_{\max}, \ldots, m_{\max}$ and $n = \max(1, |m|), \ldots, n_{\max}$. The expansion coefficients can be found separately for each azimuthal mode m and note that only two azimuthal modes ($m = \pm 1$) are required for solution construction. Therefore, the convergence of the projections schemes can be analyzed by varying the number of terms $n_{\max}$. Assuming the incident field to have unit amplitude we evaluate the normalized differential scattering cross section (DSCS) in the azimuthal plane $\varphi = 0°$. Figs. 1 and 2 show the normalized DSCS at the scattering angle $\theta = 180°$ computed with the conventional null-field method and projection schemes PS1, PS2 and PS3. The plotted data clearly demonstrate that the least-squares method has the lowest rate of convergence. In contrast, the conventional method is the most efficient since the convergence is achieved for $n_{\max} = 10$. Projection schemes PS2 and PS3 are superior to the least-squares method and actually this is due to the fact that it is possible to accelerate the convergence by decreasing the parameter λ. Recall that for $\lambda \to \infty$ the projection schemes PS2 and PS3 correspond to the least-squares

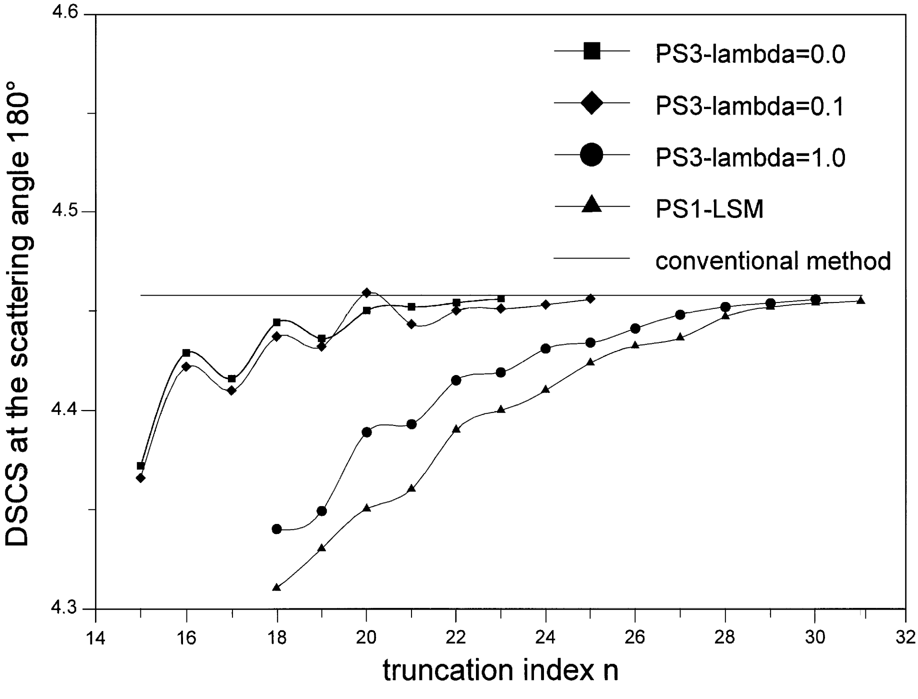

Fig. 2. The same as in Fig. 1 but the curves are now computed with the conventional null-field method and projection scheme PS3.

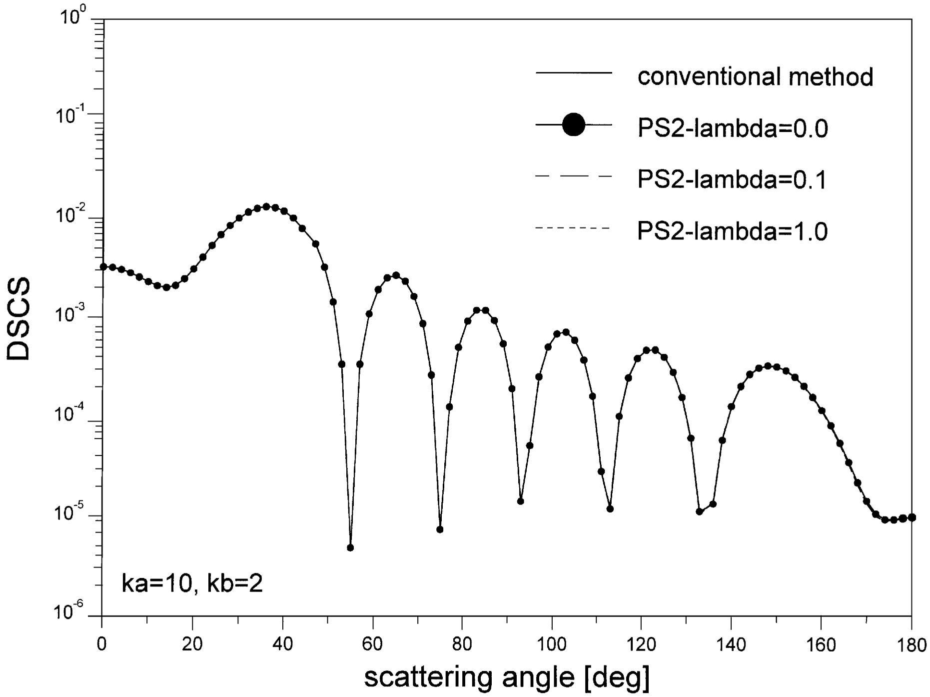

Fig. 3 . Normalized differential scattering cross section (DSCS) for a prolate spheroid with semiaxes $k_s a = 10$ and $k_s b = 2$. The curves are computed with the conventional null-field method and projection scheme PS2. The basis and testing functions are the distributed lowest-order multipoles.

method and for $\lambda = 0$ we obtain a scheme in which the surface currents are approximated by radiating spherical vector wave functions.

Let us make some remarks. In the discrete sources method the amplitudes of discrete sources may be computed by minimizing the residual fields in the least-squares sense [8,9]. However, computer simulations demonstrated that this method has a low rate of convergence. For two-dimensional geometries Yasuura mastered this problem by equipping the standard method with a smoothing procedure [10]. In this case the convergence of the numerical scheme becomes faster by increasing the order of the iterated kernel p. Therefore, the parameter λ appearing in our projection schemes is a measure of the convergence rate and plays the same role as the parameter p in the Yasuura smoothing method.

The same behavior of the solution can be observed if $\mathbf{\Psi}_\nu^{1,3}$ and $\mathbf{\Phi}_\nu^{1,3}$ are chosen as the distributed spherical vector wave functions $\mathscr{M}_{mn}^{3,1}$ and $\mathscr{N}_{mn}^{3,1}$, with $m = -m_{\max}, \dots, m_{\max}$ and $n = 1, 2, \dots, n_{\max}$. It is noted that $\mathscr{M}_{mn}^{3,1}$ and $\mathscr{N}_{mn}^{3,1}$ represent the lowest-order spherical vector wave functions having their singularities located at the particle symmetry axis. In this case $n_{\max}$ denotes the number of discrete sources.

In Fig. 3 we plot the normalized DSCS for a prolate spheroid with $k_s a = 10$ and $k_s b = 2$. The curves are computed with the conventional method and projection scheme PS2. In order to achieve convergence we use 17 sources in the first case, while 20, 24 and 27 sources corresponding to

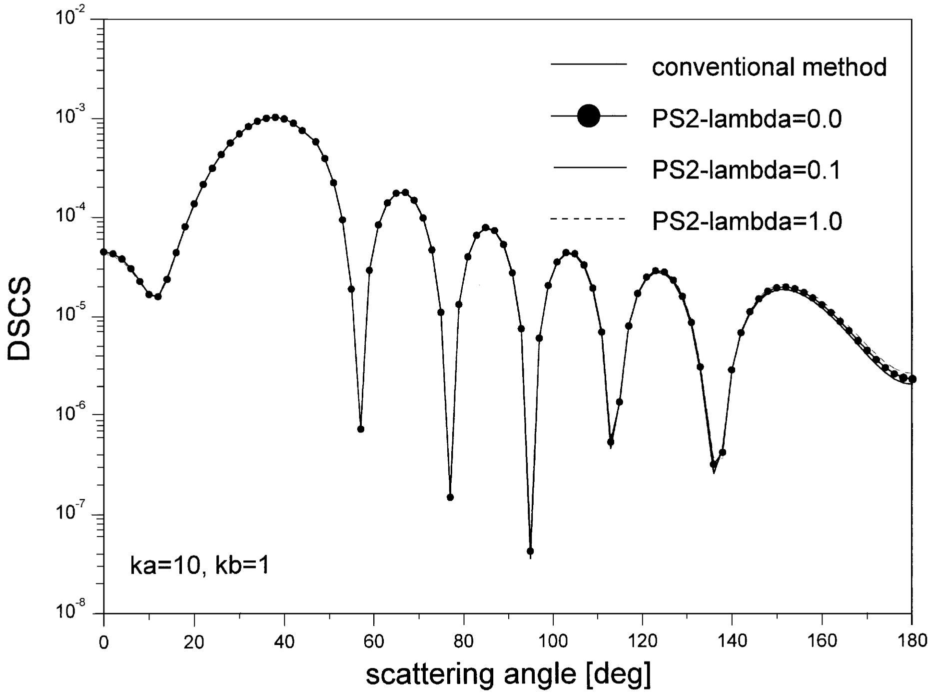

Fig. 4. Normalized differential scattering cross section (DSCS) for a prolate spheroid with semiaxes $k_s a = 10$ and $k_s b = 1$. The curves are computed with the conventional null-field method and projection scheme PS2. The basis and testing functions are the distributed lowest-order multipoles.

$\lambda = 0$, 0.1 and 1.0, respectively, are necessary in the second case. The discrepancy concerning the number of sources used is more pronounced if the aspect ratio increases. The results plotted in Fig. 4 correspond to a prolate spheroid with $k_s a = 10$ and $k_s b = 1$. As before, 17 sources leads to accurate results in the case of the conventional method, while 40, 44 and 50 discrete sources corresponding to $\lambda = 0$,0.1 and 1.0, respectively, are required in the projection scheme PS2.

4. Conclusions

In this contribution projection schemes constructed on the basis of the discrete approximation theorem are investigated. It was shown that the projection scheme corresponding to the least-squares method has a low rate of convergence. The convergence can be accelerated by using a system of vector functions consisting in a linear combination of tangential vector fields and their rotated components about the unit normal. The parameter λ appearing in these schemes can be used for controlling the convergence rate and therefore gives more flexibility to these algorithms. In addition, it was shown that the conventional null-field method is the most efficient in spite of the fact that we are not able to prove the convergence. In this context the convergence question of the conventional method is open and several addition research is needed. Our contribution demonstrate once again the geniale feeling of Waterman who chose simple derivation, preferring physical plausibility over mathematical rigor. Those who demand the latter are reminded that ‘one man’s rigor is another man’s mortis’.

Acknowledgements

The support of Volkswagen-Foundation is gratefully acknowledged.

Appendix A

In this appendix we present some fundamental results from linear functional analysis.

Let H be a Hilbert space. The mapping $\mathscr{B}: H \times H \to C$ is called a sesquilinear form on H if it is linear in the first argument and antilinear or semilinear in the second one, i.e.

$$\begin{aligned} &\mathscr{B}(\alpha x_1 + \beta x_2, y) = \alpha \mathscr{B}(x_1, y) + \beta \mathscr{B}(x_2, y), \\ &\mathscr{B}(x, \alpha y_1 + \beta y_2) = \alpha^* \mathscr{B}(x, y_1) + \beta^* \mathscr{B}(x, y_2). \end{aligned} \tag{26}$$

A sesquilinear form $\mathscr{B}$ is bounded, if it exists a real constant $M > 0$ such that

$$|\mathscr{B}(x, y)| \leq M \|x\|_H \|y\|_H, \tag{27}$$

and coercive or H-elliptic, if it exists a real constant $c > 0$ such that

$$Re\,\mathscr{B}(x, x) \geq c \|x\|_H^2. \tag{28}$$

Theorem A.1 (Fundamental Theorem of Discrete Approximation). *Let H be a Hilbert space, $\mathscr{B}$ a bounded and coercive sesquilinear form on H, $\mathscr{F}$ a linear and continuous functional on H, and $\{\psi_i\}_{i=1}^{\infty}$ a complete and linearly independent system in H. Then*

1. *the algebraic system of equations*

$$\sum_{i=1}^{N} \mathscr{B}(\psi_i, \psi_j)\, a_i^N = \mathscr{F}^*(\psi_j), \quad j = 1, \ldots, N \tag{29}$$

has an unique solution;

2. *the sequence*

$$u_N = \sum_{i=1}^{N} \alpha_i^N \psi_i \tag{30}$$

is convergent, and if $\|u_N - u\|_H \to 0$ as $N \to \infty$, then

$$\mathscr{F}^*(x) = \mathscr{B}(u, x), \quad \forall x \in H. \tag{31}$$

The proof of the above theorem can be found in [11].

Let us apply the fundamental theorem of discrete approximation to the Hilbert space $\mathscr{L}^2_{\tan}(S)$. Our aim is to construct convergent projection methods for the variational equations

$$\begin{aligned}&1.\ \langle \mathbf{u}-\mathbf{u}_0,\mathbf{x}\rangle_2 = 0, \quad \forall \mathbf{x}\in\mathscr{L}^2_{\tan}(S),\\ &2.\ \langle \mathbf{u}-\mathbf{u}_0,\mathbf{n}\times\mathbf{x}+\lambda\mathbf{x}\rangle_2 = 0, \quad \forall \mathbf{x}\in\mathscr{L}^2_{\tan}(S),\ \lambda>0,\\ &3.\ \langle (\mathbf{u}-\mathbf{u}_0)+\lambda\mathbf{n}\times(\mathbf{u}^*-\mathbf{u}_0^*),\mathbf{n}\times\mathbf{x}^*\rangle_2 = 0, \quad \forall \mathbf{x}\in\mathscr{L}^2_{\tan}(S),\ \lambda>0.\end{aligned} \tag{32}$$

Let us assume that the surface S is of class C^2. If S has the parametric representation $\mathbf{x} = \mathbf{x}(u,v)$ at each point on the surface we can define an orthogonal tangent-normal system of unit vectors $(\mathbf{e}_u, \mathbf{e}_v, \mathbf{n})$, where $\mathbf{n}$ represents the outward unit normal vector to S, and $\mathbf{e}_u$ and $\mathbf{e}_v$ are orthogonal unit vectors in the tangent plane of S.

Projection methods in $\mathscr{L}^2_{\tan}(S)$ are given by the following theorem

Theorem A.2. *Let* $\{\boldsymbol{\Psi}_i\}_{i=1}^{\infty}$ *be a complete system of vector functions in* $\mathscr{L}^2_{\tan}(S)$ *and* $\mathbf{u}_0\in\mathscr{L}^2_{\tan}(S)$. *The sequence* $\mathbf{u}_N = \sum_{i=1}^{N}\alpha_i^N\boldsymbol{\Psi}_i$, *satisfying the projection relations*

$$\begin{aligned}&1.\ \langle \mathbf{u}_N-\mathbf{u}_0,\boldsymbol{\Psi}_j\rangle_2 = 0, \quad 1\le j\le N,\\ &2.\ \langle \mathbf{u}_N-\mathbf{u}_0,\mathbf{n}\times\boldsymbol{\Psi}_j+\lambda\boldsymbol{\Psi}_j\rangle_2 = 0, \quad 1\le j\le N,\\ &3.\ \langle (\mathbf{u}_N-\mathbf{u}_0)+\lambda\mathbf{n}\times(\mathbf{u}_N^*-\mathbf{u}_0^*),\mathbf{n}\times\boldsymbol{\Psi}_j^*\rangle_2 = 0, \quad 1\le j\le N,\end{aligned} \tag{33}$$

converge in the L_2*-norm to* $\mathbf{u}_0$, i.e.

$$\|\mathbf{u}_N-\mathbf{u}_0\|_2\to 0 \quad \text{as } N\to\infty\,. \tag{34}$$

Proof. For proving 1 we define the sesquilinear form $\mathscr{B}:\mathscr{L}^2_{\tan}(S)\times\mathscr{L}^2_{\tan}(S)\to C$ by

$$\mathscr{B}(\mathbf{x},\mathbf{y}) = \langle\mathbf{x},\mathbf{y}\rangle_2, \quad \forall\mathbf{x},\mathbf{y}\in\mathscr{L}^2_{\tan}(S), \tag{35}$$

and the functional $\mathscr{F}:\mathscr{L}^2_{\tan}(S)\to C$ by

$$\mathscr{F}(\mathbf{x}) = \langle\mathbf{u}_0,\mathbf{x}\rangle_2^* = \langle\mathbf{x},\mathbf{u}_0\rangle_2, \quad \forall\mathbf{x}\in\mathscr{L}^2_{\tan}(S),\ \mathbf{u}_0\in\mathscr{L}^2_{\tan}(S). \tag{36}$$

It is obviously that $\mathscr{B}$ and $\mathscr{F}$ satisfy the requirements of the discrete approximation theorem and, hence, 1 is proved.

We consider 2. Define the sesquilinear form $\mathscr{B}:\mathscr{L}^2_{\tan}(S)\times\mathscr{L}^2_{\tan}(S)\to C$ by

$$\mathscr{B}(\mathbf{x},\mathbf{y}) = \langle\mathbf{x},\mathbf{n}\times\mathbf{y}+\lambda\mathbf{y}\rangle_2, \quad \forall\mathbf{x},\mathbf{y}\in\mathscr{L}^2_{\tan}(S), \tag{37}$$

and the linear functional $\mathscr{F}:\mathscr{L}^2_{\tan}(S)\to C$ by

$$\mathscr{F}(\mathbf{x}) = \langle\mathbf{u}_0,\mathbf{n}\times\mathbf{x}+\lambda\mathbf{x}\rangle_2^* = \langle\mathbf{n}\times\mathbf{x}+\lambda\mathbf{x},\mathbf{u}_0\rangle_2, \quad \forall\mathbf{x}\in\mathscr{L}^2_{\tan}(S). \tag{38}$$

In this context we observe that the conditions of the fundamental discrete approximation theorem are fulfilled. Indeed,

a. $\mathscr{B}$ is bounded

$$\begin{aligned}|\mathscr{B}(\mathbf{x},\mathbf{y})| &\le \|\mathbf{x}\|_2\,\|\mathbf{n}\times\mathbf{y}+\lambda\mathbf{y}\|_2 \le \|\mathbf{x}\|_2(\|\mathbf{n}\times\mathbf{y}\|_2+\|\lambda\mathbf{y}\|_2)\\ &\le (1+|\lambda|)\,\|\mathbf{x}\|_2\,\|\mathbf{y}\|_2,\end{aligned} \tag{39}$$

b. $\mathscr{B}$ is coercive

$$\begin{aligned}|\mathscr{B}(\mathbf{x},\mathbf{x})| &= |\langle \mathbf{x}, \mathbf{n}\times\mathbf{x} + \lambda\mathbf{x}\rangle_2| = |\langle \mathbf{x}, \mathbf{n}\times\mathbf{x}\rangle_2 + \langle \mathbf{x}, \lambda\mathbf{x}\rangle_2| \\ &= \left|2j\int_S \mathrm{Im}(x_v x_u^*)\,\mathrm{d}S + \lambda\int_S |\mathbf{x}|^2\,\mathrm{d}S\right| \geq |\lambda|\,\|\mathbf{x}\|_2^2,\end{aligned} \tag{40}$$

where $\mathbf{x} = x_u\mathbf{e}_u + x_v\mathbf{e}_v$, and

c. $\mathscr{F}$ is continuous

$$|\mathscr{F}(\mathbf{x})| = |\langle \mathbf{n}\times\mathbf{x} + \lambda\mathbf{x}, \mathbf{u}_0\rangle_2| \leq (1 + |\lambda|)\,\|\mathbf{x}\|_2\,\|\mathbf{u}_0\|_2. \tag{41}$$

Consequently, if $\{\mathbf{\Psi}_i\}_{i=1}^{\infty}$ is a complete system of vector functions in $\mathscr{L}^2_{\tan}(S)$, the sequence $\mathbf{u}_N = \sum_{i=1}^{N}\alpha_i^N\mathbf{\Psi}_i$ satisfying the algebraic system of equations

$$\mathscr{B}(\mathbf{u}_N, \mathbf{\Psi}_j) = \mathscr{F}^*(\mathbf{\Psi}_j), \quad 1 \leq j \leq N \Leftrightarrow \langle \mathbf{u}_N - \mathbf{u}_0, \mathbf{n}\times\mathbf{\Psi}_j + \lambda\mathbf{\Psi}_j\rangle_2 = 0, \quad 1 \leq j \leq N, \tag{42}$$

converges in the L_2-norm to the unique solution $\mathbf{u}$ of the variational equation

$$\mathscr{B}(\mathbf{u},\mathbf{x}) = \mathscr{F}^*(\mathbf{x}), \quad \forall\mathbf{x}\in\mathscr{L}^2_{\tan}(S) \Leftrightarrow \langle \mathbf{u} - \mathbf{u}_0, \mathbf{n}\times\mathbf{x} + \lambda\mathbf{x}\rangle_2 = 0, \quad \forall\mathbf{x}\in\mathscr{L}^2_{\tan}(S). \tag{43}$$

Note that the uniqueness of the solution to the variational equation (43) follows from a corollary of Lax–Milgram Lemma [11]. We observe that for any given $\mathbf{y}\in\mathscr{L}^2_{\tan}(S)$, $\mathbf{y} = y_u\mathbf{e}_u + y_v\mathbf{e}_v$, we find an unique tangential vector function $\mathbf{x}\in\mathscr{L}^2_{\tan}(S)$, $\mathbf{x} = x_u\mathbf{e}_u + x_v\mathbf{e}_v$, with the property $\mathbf{y}=\mathbf{n}\times\mathbf{x} + \lambda\mathbf{x}$, that is $x_u = (\lambda y_u + y_v)/(1 + \lambda^2)$ and $x_v = (\lambda y_v - y_u)/(1 + \lambda^2)$. Thus, the unique solution of the variational equation (43) is obtained by choosing $\mathbf{x}\in\mathscr{L}^2_{\tan}(S)$ such that $\mathbf{u} - \mathbf{u}_0 = \mathbf{n}\times\mathbf{x} + \lambda\mathbf{x}$. We receive $\mathbf{u} = \mathbf{u}_0$ in $\mathscr{L}^2_{\tan}(S)$ and 2 is proved.

We pass to 3. First, we prove the unique solvability of the corresponding system of equations. It is shown that for $\mathbf{u}_N = \sum_{i=1}^{N}\alpha_i^N\mathbf{\Psi}_i$ from

$$\langle \mathbf{u}_N + \lambda\mathbf{n}\times\mathbf{u}_N^*, \mathbf{n}\times\mathbf{\Psi}_j^*\rangle_2 = 0, \quad 1 \leq j \leq N \tag{44}$$

we receive $\alpha_i^N = 0$, $i = 1, 2, \ldots, N$. Multiply each equation in (44) by α_j^N and sum the resulting expressions. We get

$$\langle \mathbf{u}_N + \lambda\mathbf{n}\times\mathbf{u}_N^*, \mathbf{n}\times\mathbf{u}_N^*\rangle_2 = 0. \tag{45}$$

Consequently, $\|\mathbf{u}_N\|_2 = 0$, and since the system $\{\mathbf{\Psi}_i\}_{i=1}^{N}$ is linear independent on S the conclusion follows.

Let us prove the convergence. From the completeness property of the system $\{\mathbf{\Psi}_i\}_{i=1}^{\infty}$ we deduce that for a given $\mathbf{u}_0\in\mathscr{L}^2_{\tan}(S)$ there exist a sequence $\mathscr{U}_N = \sum_{i=1}^{N}\beta_i^N\mathbf{\Psi}_i$ such that $\|\boldsymbol{\varepsilon}_N\|_2 \to 0$ as $N \to \infty$. Here $\boldsymbol{\varepsilon}_N = \mathscr{U}_N - \mathbf{u}_0$ represents the approximation error on S. Then, we have

$$\langle (\mathscr{U}_N - \mathbf{u}_0) + \lambda\mathbf{n}\times(\mathscr{U}_N^* - \mathbf{u}_0^*), \mathbf{n}\times\mathbf{\Psi}_j^*\rangle_2 = \langle \boldsymbol{\varepsilon}_N + \lambda\mathbf{n}\times\boldsymbol{\varepsilon}_N^*, \mathbf{n}\times\mathbf{\Psi}_j^*\rangle_2, \quad 1 \leq j \leq N. \tag{46}$$

Subtracting (A.8-3) from (46) we receive

$$\langle \delta\mathbf{u}_N + \lambda\mathbf{n}\times\delta\mathbf{u}_N^*, \mathbf{n}\times\mathbf{\Psi}_j^*\rangle_2 = \langle \boldsymbol{\varepsilon}_N + \lambda\mathbf{n}\times\boldsymbol{\varepsilon}_N^*, \mathbf{n}\times\mathbf{\Psi}_j^*\rangle_2, \quad 1 \leq j \leq N \tag{47}$$

with $\delta\mathbf{u}_N = \mathscr{U}_N - \mathbf{u}_N$. Since $\delta\mathbf{u}_N$ can be expressed as a linear combination of $\mathbf{\Psi}_i$ with $i = 1, 2, \ldots, N$, we find that

$$\langle \delta\mathbf{u}_N + \lambda\mathbf{n} \times \delta\mathbf{u}_N^*, \mathbf{n} \times \delta\mathbf{u}_N^* \rangle_2 = \langle \boldsymbol{\varepsilon}_N + \lambda\mathbf{n} \times \boldsymbol{\varepsilon}_N^*, \mathbf{n} \times \delta\mathbf{u}_N^* \rangle_2. \tag{48}$$

Therefore, from

$$\begin{aligned} |\lambda| \|\delta\mathbf{u}_N\|_2^2 &= |\langle \delta\mathbf{u}_N + \lambda\mathbf{n} \times \delta\mathbf{u}_N^*, \mathbf{n} \times \delta\mathbf{u}_N^* \rangle_2| \\ &= |\langle \boldsymbol{\varepsilon}_N + \lambda\mathbf{n} \times \boldsymbol{\varepsilon}_N^*, \mathbf{n} \times \delta\mathbf{u}_N^* \rangle_2| \le (1 + |\lambda|) \|\boldsymbol{\varepsilon}_N\|_2 \|\delta\mathbf{u}_N\|_2 \end{aligned} \tag{49}$$

we get

$$\|\delta\mathbf{u}_N\|_2 \le \frac{(1 + |\lambda|)}{|\lambda|} \|\boldsymbol{\varepsilon}_N\|_2, \tag{50}$$

and the conclusion $\|\mathbf{u}_N - \mathbf{u}_0\|_2 \to 0$ as $N \to \infty$ follows immediately. □

Let us make some comments.

1. The algebraic system of equations corresponding to (A.8-1) is the Gramm system of the linearly independent system $\{\mathbf{\Psi}_i\}_{i=1}^N$, and $\mathbf{u}_N$ is the projection of $\mathbf{u}$ onto the N-dimensional space $H_N = Sp\{\mathbf{\Psi}_1, \ldots, \mathbf{\Psi}_N\}$, i.e. $P_N\mathbf{u} = \mathbf{u}_N$.

2. Projection relations (A.8-2) can be regarded as closeness relations for the complete system $\{\mathbf{\Phi}_i\}_{i=1}^\infty$, where $\mathbf{\Phi}_i = \mathbf{n} \times \mathbf{\Psi}_i + \lambda\mathbf{\Psi}_i$. Indeed, let $\mathbf{x}, \mathbf{y} \in \mathscr{L}^2_{\tan}(S)$, such that $\mathbf{y} = \mathbf{n} \times \mathbf{x} + \lambda\mathbf{x}$. Since $\{\mathbf{\Psi}_i\}_{i=1}^\infty$ is a complete system in $\mathscr{L}^2_{\tan}(S)$ we find $\mathbf{x}_N = \sum_{i=1}^N \alpha_i^N \mathbf{\Psi}_i$, such that $\|\mathbf{x}_N - \mathbf{x}\|_2 \to 0$ and $\|\mathbf{n} \times \mathbf{x}_N - \mathbf{n} \times \mathbf{x}\|_2 \to 0$ as $N \to \infty$. Consequently, $\|\mathbf{y} - (\mathbf{n} \times \mathbf{x}_N + \lambda\mathbf{x}_N)\|2 \to 0$ as $N \to \infty$ and we see that the system $\{\mathbf{\Phi}_i\}_{i=1}^\infty$ is complete in $\mathscr{L}^2_{\tan}(S)$. The discrete approximation theorem can also be applied with the sesquilinear form $\mathscr{B}: \mathscr{L}^2_{\tan}(S) \times \mathscr{L}^2_{\tan}(S) \to C$ defined by

$$\mathscr{B}(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x} + \lambda\mathbf{n} \times \mathbf{x}, \mathbf{n} \times \mathbf{y} \rangle_2, \quad \forall \mathbf{x}, \mathbf{y} \in \mathscr{L}^2_{\tan}(S), \tag{51}$$

and the linear functional $\mathscr{F}: \mathscr{L}^2_{\tan}(S) \to C$ defined by

$$\mathscr{F}(\mathbf{x}) = \langle \mathbf{u}_0 + \lambda\mathbf{n} \times \mathbf{u}_0, \mathbf{n} \times \mathbf{x} \rangle_2^* = \langle \mathbf{n} \times \mathbf{x}, \mathbf{u}_0 + \lambda\mathbf{n} \times \mathbf{u}_0 \rangle_2, \quad \forall \mathbf{x} \in \mathscr{L}^2_{\tan}(S). \tag{52}$$

In this case, it follows that the sequence $\mathbf{u}_N = \sum_{i=1}^N \alpha_i^N \mathbf{\Psi}_i$, satisfying the projection relations

$$\langle (\mathbf{u}_N - \mathbf{u}_0) + \lambda\mathbf{n} \times (\mathbf{u}_N - \mathbf{u}_0), \mathbf{n} \times \mathbf{\Psi}_j \rangle_2 = 0, \quad 1 \le j \le N \tag{53}$$

converges in the L_2-norm to $\mathbf{u}_0$. Thus, the sequence $\mathbf{u}'_N$,

$$\mathbf{u}'_N = \mathbf{u}_N + \lambda\mathbf{n} \times \mathbf{u}_N = \sum_{i=1}^N \alpha_i^N (\mathbf{\Psi}_i + \lambda\mathbf{n} \times \mathbf{\Psi}_i) \tag{54}$$

satisfying the projection relations

$$\langle \mathbf{u}'_N - \mathbf{u}'_0, \mathbf{n} \times \mathbf{\Psi}_j \rangle_2 = 0, \quad 1 \le j \le N \tag{55}$$

converges in the L_2-norm to $\mathbf{u}'_0 = \mathbf{u}_0 + \lambda\mathbf{n} \times \mathbf{u}_0$.

3. According to part 3 of Theorem A.2 we see that the sequence $\mathbf{u}'_N$,

$$\mathbf{u}'_N = \mathbf{u}_N + \lambda\mathbf{n} \times \mathbf{u}_N^* = \sum_{i=1}^N \alpha_i^N \mathbf{\Psi}_i + \alpha_i^{N*} (\lambda\mathbf{n} \times \mathbf{\Psi}_i^*) \tag{56}$$

satisfying the projection relations

$$\langle \mathbf{u}'_N - \mathbf{u}'_0, \mathbf{n} \times \boldsymbol{\Psi}_j^* \rangle_2 = 0, \quad 1 \le j \le N \tag{57}$$

converge in the L_2-norm to $\mathbf{u}'_0 = \mathbf{u}_0 + \lambda \mathbf{n} \times \mathbf{u}_0^*$.

Appendix B

In this appendix we give a counterexample showing that the spherical vector wave functions do not form a basis in $\mathscr{L}^2_{\tan}(S)$. Consider a spherical surface $S^{\mathrm{r}} = \partial D_i^{\mathrm{r}}$ dividing S into exactly two parts: the first one S_1 is in the exterior of D_i^{r}, and the second one S_2 is contained in D_i^{r}. Define the field $\mathscr{E}(\mathbf{x}) = \nabla \times [\mathbf{a} g(\mathbf{x}, \mathbf{y}, k_{\mathrm{s}})]$, where $\mathbf{a}$ is a constant vector and $\mathbf{y}$ is fixed point on S^{r}. Using the vector spherical multipole expansion of the Green function

$$g(\mathbf{x}, \mathbf{y}, k_{\mathrm{s}})\overset{=}{\mathbf{I}} = \frac{jk_{\mathrm{s}}}{\pi} \sum_{\mu=1}^{\infty} D_\mu \begin{cases} \mathbf{M}^3_{\bar{\mu}}(k_{\mathrm{s}}\mathbf{y})\mathbf{M}^1_\mu(k_{\mathrm{s}}\mathbf{x}) + \mathbf{N}^3_{\bar{\mu}}(k_{\mathrm{s}}\mathbf{y})\mathbf{N}^1_\mu(k_{\mathrm{s}}\mathbf{x}) \\ \mathbf{M}^1_{\bar{\mu}}(k_{\mathrm{s}}\mathbf{y})\mathbf{M}^3_\mu(k_{\mathrm{s}}\mathbf{x}) + N^1_{\bar{\mu}}(k_{\mathrm{s}}\mathbf{y})N^3_\mu(k_{\mathrm{s}}\mathbf{x}) \end{cases} + \mathit{irrotational\ terms} \begin{cases} y > x, \\ y < x, \end{cases} \tag{58}$$

we obtain for $\mathbf{x} \in S_1$, $x > y$,

$$\mathbf{n}(\mathbf{x}) \times \mathscr{E}(\mathbf{x}) = \sum_{\mu=1}^{\infty} a_\mu [\mathbf{n}(\mathbf{x}) \times \mathbf{M}^3_\mu(k_{\mathrm{s}}\mathbf{x})] + b_\mu [\mathbf{n}(\mathbf{x}) \times \mathbf{N}^3_\mu(k_{\mathrm{s}}\mathbf{x})], \tag{59}$$

where

$$a_\mu = \frac{jk_{\mathrm{s}}^2}{\pi} D_\mu (\mathbf{a} \cdot \mathbf{N}^1_{\bar{\mu}}(k_{\mathrm{s}}\mathbf{y})) \quad \text{and} \quad b_\mu = \frac{jk_{\mathrm{s}}^2}{\pi} D_\mu \left(\mathbf{a} \cdot \mathbf{M}^1_{\bar{\mu}}(k_{\mathrm{s}}\,\mathbf{y}) \right), \tag{60}$$

and D_μ is a normalization constant. Assume that (59) is also valid for $\mathbf{x} \in S_2$ for which $x < y$. Then, this contradicts the fact that the valid representation of $\mathbf{n} \times \mathscr{E}$ on S_2, is

$$\mathbf{n}(\mathbf{x}) \times \mathscr{E}(\mathbf{x}) = \sum_{\mu=1}^{\infty} a'_\mu [\mathbf{n}(\mathbf{x}) \times \mathbf{M}^1_\mu(k_{\mathrm{s}}\mathbf{x})] + b'_\mu [\mathbf{n}(\mathbf{x}) \times \mathbf{N}^1_\mu(k_{\mathrm{s}}\mathbf{x})], \tag{61}$$

where

$$a'_\mu = \frac{jk_{\mathrm{s}}^2}{\pi} D_\mu \left(\mathbf{a} \cdot \mathbf{N}^3_{\bar{\mu}}(k_{\mathrm{s}}\,\mathbf{y}) \right) \quad \text{and} \quad b'_\mu = \frac{jk_{\mathrm{s}}^2}{\pi} D_\mu (\mathbf{a} \cdot \mathbf{M}^3_{\bar{\mu}}(k_{\mathrm{s}}\,\mathbf{y})) \tag{62}$$

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