Computer Physics Communications 124 (2000) 19–27

Computer Physics Communications

www.elsevier.nl/locate/cpc

T- and D-matrix methods for electromagnetic scattering by impedance obstacles

A. Doicu [a,*], Yu.A. Eremin [b], T. Wriedt [a]

[a] *Institut für Werkstofftechnik, Badgasteiner Straße 3, 28359 Bremen, Germany*

[b] *Department of Applied Mathematics and Computer Science, Moscow State University, Vorobyov Hills, 119899 Moscow, Russia*

Received 7 June 1999

Abstract

The T-matrix method and a new projection method for the impedance boundary value problem in electromagnetic scattering theory are presented. The new method is called the D-matrix method since the matrix of the linear system of equations is dissipative. The dissipativity is established as a consequence of the conservation law of energy. The convergence and solvability of the linear system of equations appearing in the D-matrix method is validated. Numerical experiments are performed for analyzing the rate of convergence of both methods. The numerical analysis indicates that the T-matrix method still stands tall among alternatives approaches. © 2000 Elsevier Science B.V. All rights reserved.

1. Introduction

Three-dimensional problems of electromagnetic scattering have been the subject of intense investigation and research. One of the fastest and most powerful numerical tools for computing the nonspherical light scattering is the T-matrix method [1]. In this method, a set of integral equations for the surface current densities is derived by considering the null-field condition for the scattered field inside the particle. An approximate solution of the scattering problem is then obtained by approximating the surface fields by the complete set of tangential single spherical coordinate vector wave functions. The theoretical foundations of the method were given by Ramm [2], and Kristensson, Ramm and Ström [3]. They include analysis of the convergence of the method, stability of the numerical scheme towards small perturbations of data and estimates of rate of convergence. Remarkably enough, particular satisfactory criteria for choosing the complete family of functions to approximate the surface current densities were given. According to these criteria the single spherical coordinate vector wave functions can not be used for surface current approximation since in this case there are no guarantees that the scheme converges.

For particles with extreme geometries or particles with appreciable concavities the single spherical coordinate based null-field method fails to converge. A number of modifications to the conventional T-matrix method have been suggested to improve the numerical stability. One of these formal modifications is the T-matrix method with discrete sources [4]. Essentially, this method entails the use of a number of elementary sources for approximating the surface current densities. Unknown discrete sources amplitudes which produces the surface densities are computed by using the null-field condition of the total electric field inside the particle surface.

* Corresponding author. E-mail: doicu@iwt.uni-bremen.de.

PII: S0010-4655(99)00426-9

The subject matter of this paper is to modify the T-matrix scheme in order to construct a convergent scheme for scattered field determination. The new method is called the D-matrix method since the matrix of the projective scheme is dissipative. We prove the convergence and unique solvability of the linear system of equations for any system of functions which are complete on the particle surface. In the present paper we extend the treatment of acoustic wave propagation in the presence of an impedance obstacle to the electromagnetic case [5].

2. T-matrix method

Let us consider the scattering problem of a given incident field on a local obstacle D in R^3 with the impedance boundary condition on the surface $S = \partial D$. The boundary-value problem consists in Maxwell equations in $R^3 - \overline{D}$

$$\nabla \times \mathbf{E}_s = jk\mathbf{H}_s,$$
$$\nabla \times \mathbf{H}_s = -jk\mathbf{E}_s, \tag{1}$$

the impedance boundary condition on S

$$\mathbf{n} \times \mathbf{E}_s - \lambda\mathbf{n} \times (\mathbf{n} \times \mathbf{H}_s) = \mathbf{f}, \tag{2}$$

and the radiation condition

$$\lim_{r\to\infty} r(\mathbf{H}_s \times \mathbf{e}_r - \mathbf{E}_s) = \mathbf{0} \quad \text{as } r \to \infty, \tag{3}$$

uniformly for all radial directions. Here, k is the wave number, $k > 0$ by assumption,

$$\mathbf{f} = -\big[\mathbf{n} \times \mathbf{E}_0 - \lambda\mathbf{n} \times (\mathbf{n} \times \mathbf{H}_0)\big], \tag{4}$$

where is $\mathbf{E}_0$, $\mathbf{H}_0$ is the incident field and λ is the given impedance function with the property $\operatorname{Re}\lambda \geqslant 0$. According to [6] there exists an unique solution of the boundary-value problem (1)–(3).

For solving the impedance boundary-value problem in the framework of the T-matrix method with discrete sources the scattering object is replaced by a set of surface current densities, so that in the exterior domain the sources and fields are exactly the same as those existing in the original scattering problem. The entire analysis can conveniently be broken down into the following three steps:

(1) A set of integral equations for the surface current density is derived for a variety of discrete sources. Physically, the set of integral equations in question guarantees the null-field condition within D. Essentially, the T-matrix method with discrete sources consists in the projection relations:

$$-\frac{k^2}{\pi}\int_S (\mathbf{n} \times \mathbf{H}) \cdot \big(\mathbf{\Psi}_\nu^3 + j\lambda\mathbf{n} \times \mathbf{\Phi}_\nu^3\big)\, dS = a_\nu,$$
$$-\frac{k^2}{\pi}\int_S (\mathbf{n} \times \mathbf{H}) \cdot \big(\mathbf{\Phi}_\nu^3 + j\lambda\mathbf{n} \times \mathbf{\Psi}_\nu^3\big)\, dS = b_\nu, \tag{5}$$

for $\nu = 1, 2, \ldots$, where

$$a_\nu = j\frac{k^2}{\pi}\int_S \big[(\mathbf{n} \times \mathbf{E}_0) \cdot \mathbf{\Phi}_\nu^3 + j(\mathbf{n} \times \mathbf{H}_0) \cdot \mathbf{\Psi}_\nu^3\big]\, dS,$$
$$b_\nu = j\frac{k^2}{\pi}\int_S \big[(\mathbf{n} \times \mathbf{E}_0) \cdot \mathbf{\Psi}_\nu^3 + j(\mathbf{n} \times \mathbf{H}_0) \cdot \mathbf{\Phi}_\nu^3\big]\, dS, \tag{6}$$

and $\mathbf{E}$, $\mathbf{H}$ stands for the total field, i.e. $\mathbf{E} = \mathbf{E}_s + \mathbf{E}_0$ and $\mathbf{H} = \mathbf{H}_s + \mathbf{H}_0$. The set $\{\mathbf{\Psi}_\nu^{1,3}, \mathbf{\Phi}_\nu^{1,3}\}_{\nu=1,2,\ldots}$ depends on the system of discrete sources which is used for imposing the null-field condition. Actually, it stands for the sets of:

– localized vector spherical functions $\{\mathbf{M}_{mn}^{1,3}, \mathbf{N}_{mn}^{1,3}\}_{m\in\mathbf{Z}, n\geqslant\max(1,|m|)}$,

– distributed vector spherical functions $\{\mathcal{M}_{mn}^{1,3}, \mathcal{N}_{mn}^{1,3}\}_{m\in\mathbf{Z}, n=1,2,\ldots}$:

$$\mathcal{M}_{mn}^{1,3}(\mathbf{r}) = \mathbf{M}_{m,|m|+l}^{1,3}(\mathbf{r} - z_n\mathbf{e}_3),$$
$$\mathbf{r} \in \mathbf{R}^3 - \{z_n\mathbf{e}_3\}_{n=1}^{\infty},$$
$$\mathcal{N}_{mn}^{1,3}(\mathbf{r}) = \mathbf{N}_{m,|m|+l}^{1,3}(\mathbf{r} - z_n\mathbf{e}_3),$$
$$\mathbf{r} \in \mathbf{R}^3 - \{z_n\mathbf{e}_3\}_{n=1}^{\infty}, \tag{7}$$

where $l = 1$ if $m = 0$ and $l = 0$ if $m \neq 0$, and $\{z_n\}_{n=1}^{\infty}$ is a set of points located on a segment Γ_z of the z-axis, and

– magnetic and electric dipoles $\{\mathcal{M}_{ni}^{1,3}, \mathcal{N}_{ni}^{1,3}\}_{n=1,2,\ldots,\, i=1,2}$:

$$\mathcal{M}_{ni}^{1,3}(\mathbf{r}) = \mathbf{m}\big(\mathbf{r}_n^{\pm}, \mathbf{r}, \tau_{ni}^{\pm}\big), \quad \mathbf{r} \in \mathbf{R}^3 - \big\{\mathbf{r}_n^{\pm}\big\}_{n=1}^{\infty},$$
$$\mathcal{N}_{ni}^{1,3}(\mathbf{r}) = \mathbf{n}\big(\mathbf{r}_n^{\pm}, \mathbf{r}, \boldsymbol{\tau}_{ni}^{\pm}\big), \quad \mathbf{r} \in \mathbf{R}^3 - \big\{\mathbf{r}_n^{\pm}\big\}_{n=1}^{\infty}, \tag{8}$$

where $\boldsymbol{\tau}_{n1}$ and $\boldsymbol{\tau}_{n2}$ are two tangential linear independent unit vectors at the point $\mathbf{x}_n$,

$$\mathbf{m}(\mathbf{r}, \mathbf{r}', \mathbf{a}) = \frac{1}{k^2}\mathbf{a}(\mathbf{r}) \times \nabla' g(\mathbf{r}, \mathbf{r}', k),$$
$$\mathbf{n}(\mathbf{r}, \mathbf{r}', \mathbf{a}) = \frac{1}{k}\nabla' \times \mathbf{m}(\mathbf{r}, \mathbf{r}', \mathbf{a}), \quad \mathbf{r} \neq \mathbf{r}', \tag{9}$$

and the sequence $\{\mathbf{r}_n^-\}_{n=1}^{\infty}$ is dense on a smooth surface S^- enclosed in D, while the sequence $\{\mathbf{r}_n^+\}_{n=1}^{\infty}$ is dense on a smooth surface S^+ enclosing D.

We mention that the unique solvability of the infinite system (5) follows from the completeness of the system

$$\big\{\mathbf{n}\times\mathbf{\Psi}_\nu^3 + j\lambda\mathbf{n}\times\big(\mathbf{n}\times\mathbf{\Phi}_\nu^3\big),\ \mathbf{n}\times\mathbf{\Phi}_\nu^3 + j\lambda\mathbf{n}\times\big(\mathbf{n}\times\mathbf{\Psi}_\nu^3\big)\big\}_{\nu=1}^{\infty}$$

in $\mathcal{L}^2_{\tan}(S)$. This assertion is proved in Appendix A for localized spherical vector wave functions and the same technique can be used to prove the completeness of distributed spherical vector wave functions and magnetic and electric dipoles.

(2) The surface current densities are approximated by fields of discrete sources. In this context, assuming that the system $\{\mathbf{n}\times\mathbf{\Psi}_\mu^1, \mathbf{n}\times\mathbf{\Phi}_\mu^1\}_{\mu=1}^{\infty}$ is complete in $\mathcal{L}^2_{\tan}(S)$ we represent the approximate surface current densities as

$$\mathbf{n}\times\mathbf{H}_N = -j\sum_{\mu=1}^{N} a_\mu^N\mathbf{n}\times\mathbf{\Phi}_\mu^1 + b_\mu^N\mathbf{n}\times\mathbf{\Psi}_\mu^1. \tag{10}$$

(3) Once the surface current densities are determined the approximate scattered field outside the circumscribing sphere is obtained by using the representation theorem. We get

$$\mathbf{E}_{sN} = \sum_{\nu=1}^{N} f_\nu^N\mathbf{M}_\nu^3 + g_\nu^N\mathbf{N}_\nu^3, \tag{11}$$

where

$$f_\nu^N = -\frac{k^2}{\pi}\int_S (\mathbf{n}\times\mathbf{H}_N)\cdot\big(\mathbf{M}_{\overline{\nu}}^1 + j\lambda\mathbf{n}\times\mathbf{N}_{\overline{\nu}}^1\big)\,dS,$$

$$g_\nu^N = -\frac{k^2}{\pi}\int_S (\mathbf{n}\times\mathbf{H}_N)\cdot\big(\mathbf{N}_{\overline{\nu}}^1 + j\lambda\mathbf{n}\times\mathbf{M}_{\overline{\nu}}^1\big)\,dS. \tag{12}$$

Here, $\overline{\nu}$ is a complex index incorporating $-m$ and n, i.e. $\overline{\nu} = (-m, n)$.

Mathematical justification of the T-matrix scheme requires positive answers to the following questions:

(1) Is the truncated system (5) solvable for sufficiently large N?
(2) Does $\|\mathbf{n}\times\mathbf{H}_N - \mathbf{n}\times\mathbf{H}\|_{2,S} \to 0$ as $N\to\infty$?
(3) Is the truncated system (5) correctly solvable for any finite N?

As shown by Kristensson et al. [3] the answer to the above questions is yes if the complete sets for surface currents approximation and for imposing the null-field conditions form a Riesz basis in $\mathcal{L}^2_{\tan}(S)$. However, the system of spherical vector wave functions do not form a Riesz basis on S, unless S is a sphere. This conclusion follows from the fact that the condition number for the truncated Gramm matrix of the multipole system grows to infinity as the truncation size grows to infinity.

3. D-matrix method

In this section we shall summarize the scheme of the D-matrix method for the impedance boundary-value problem. Let us construct an approximation of the scattered field as

$$\mathbf{E}_{sN} = \sum_{\mu=1}^{N} a_\mu^N\mathbf{\Psi}_\mu^3 + b_\mu^N\mathbf{\Phi}_\mu^3,$$

$$\mathbf{H}_{sN} = -j\sum_{\mu=1}^{N} a_\mu^N\mathbf{\Phi}_\mu^3 + b_\mu^N\mathbf{\Psi}_\mu^3, \tag{13}$$

and determine the coefficients a_μ^N and b_μ^N from the following projection relations

$$\big\langle\mathbf{n}\times\mathbf{E}_{sN} - \lambda\mathbf{n}\times(\mathbf{n}\times\mathbf{H}_{sN}) - \mathbf{f}, \mathbf{\Phi}_\nu^3\big\rangle_{2,S} = 0,$$

$$\big\langle\mathbf{n}\times\mathbf{E}_{sN} - \lambda\mathbf{n}\times(\mathbf{n}\times\mathbf{H}_{sN}) - \mathbf{f}, \mathbf{\Psi}_\nu^3\big\rangle_{2,S} = 0, \tag{14}$$

where $\nu = 1, 2, \ldots$ and $\langle.,.\rangle_{2,S}$ stands for the scalar product in $\mathcal{L}^2_{\tan}(S)$. Then, we obtain the following system of equations for amplitudes determination

$$\sum_{\mu=1}^{N} D_{\nu\mu}^{11}a_\mu^N + D_{\nu\mu}^{12}b_\mu^N = \int_S \mathbf{f}\cdot\mathbf{\Phi}_\nu^{3*}\,dS,$$

$$\sum_{\mu=1}^{N} D_{\nu\mu}^{21}a_\mu^N + D_{\nu\mu}^{22}b_\mu^N = \int_S \mathbf{f}\cdot\mathbf{\Psi}_\nu^{3*}\,dS, \tag{15}$$

where

$$D_{\nu\mu}^{11} = \int_S \big[\mathbf{n}\times\mathbf{\Psi}_\mu^3 + j\lambda\mathbf{n}\times(\mathbf{n}\times\mathbf{\Phi}_\mu^3)\big]\cdot\mathbf{\Phi}_\nu^{3*}\,dS,$$

$$D_{\nu\mu}^{12} = \int_S \big[\mathbf{n}\times\mathbf{\Phi}_\mu^3 + j\lambda\mathbf{n}\times(\mathbf{n}\times\mathbf{\Psi}_\mu^3)\big]\cdot\mathbf{\Phi}_\nu^{3*}\,dS,$$

$$D_{\nu\mu}^{21} = \int_S \left[\mathbf{n} \times \mathbf{\Psi}_\mu^3 + j\lambda\mathbf{n}\times(\mathbf{n} \times \mathbf{\Phi}_\mu^3)\right] \cdot \mathbf{\Psi}_\nu^{3*} dS,$$

$$D_{\nu\mu}^{22} = \int_S \left[\mathbf{n} \times \mathbf{\Phi}_\mu^3 + j\lambda\mathbf{n}\times(\mathbf{n} \times \mathbf{\Psi}_\mu^3)\right] \cdot \mathbf{\Psi}_\nu^{3*} dS. \quad (16)$$

In Appendix B we prove the unique solvability of the system (15) and the fact that

$$\lim_{N\to\infty} \|\mathbf{E}_{s0} - \mathbf{E}_{sN0}\|_{2,\Omega} = 0, \quad (17)$$

where $\mathbf{E}_{s0}$ and $\mathbf{E}_{sN0}$ are the far-field patterns of the exact and approximate solutions $\mathbf{E}_s$, $\mathbf{H}_s$, and $\mathbf{E}_{sN}$, $\mathbf{H}_{sN}$, respectively, and Ω is the unit sphere. Therefore, from a theoretical point of view, the D-matrix method leads to convergent results for any system of functions which is complete on the particle surface.

4. Numerical simulations

In this section we present some computer simulations in order to give a clear picture on the convergence of the above projection schemes.

In our first example we consider a prolate spheroid with semiaxes $ka = 2$ and $kb = 1$. The incident field is a plane wave propagating along the particle symmetry axis. The surface current density is approximated by linear combinations of localized spherical vector wave functions. The expansion coefficients can be found separately for each azimuthal mode m and note that only two azimuthal modes ($m = \pm 1$) are required for solution construction. Therefore, the convergence of the projections schemes can be analyzed by varying the number of terms $n_{\max}$. Assuming the incident field to have unit amplitude we evaluate the normalized differential scattering cross section (DSCS) in the azimuthal plane $\varphi = 0°$. Figs. 1 and 2 show the normalized DSCS at the scattering angle $\theta = 180°$ computed with the T-matrix method and D-matrix method. The plotted data clearly demonstrate that the D-matrix method has the lowest rate of convergence.

The same behavior of the solution can be observed if $\mathbf{\Psi}_\nu^{1,3}$ and $\mathbf{\Phi}_\nu^{1,3}$ are chosen as the distributed spherical vector wave functions $\mathcal{M}_{mn}^{3,1}$ and $\mathcal{N}_{mn}^{3,1}$. In this case $n_{\max}$ denotes the number of discrete sources. Figs. 3 and 4 show the normalized DSCS at the scattering angle $\theta = 180°$ for a prolate spheroid with semiaxes $ka = 10$ and $kb = 2$ and two values of λ, $\lambda = 10$ and $\lambda = 2$. In addition to the conventional T-matrix method we used an approach in which the surface currents are approximated by radiating functions, i.e.

$$\mathbf{n} \times \mathbf{H}_N = -j \sum_{\mu=1}^{N} a_\mu^N \mathbf{n} \times \mathbf{\Phi}_\mu^3 + b_\mu^N \mathbf{n} \times \mathbf{\Psi}_\mu^3. \quad (18)$$

As before, the T-matrix method which uses regular functions for surface current densities approximations is the most efficient. In Fig. 5 we plot the normalized DSCS for a prolate spheroid with $ka = 10$ and $kb = 2$. The curves are computed with the conventional T-

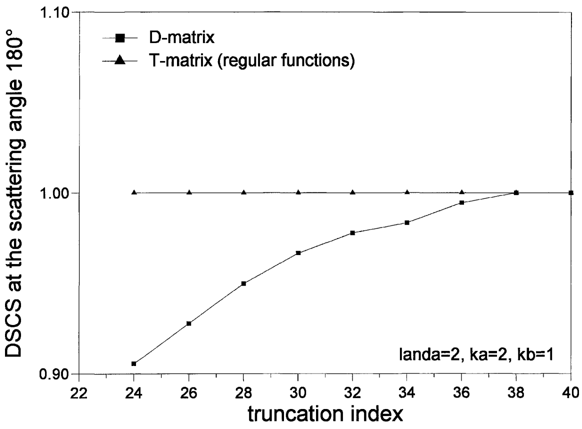

Fig. 1. Normalized differential scattering cross section (DSCS) at the scattering angle $\theta = 180°$ for different values of the truncation index. The scatterer is a prolate spheroid with $\lambda = 2$ and semiaxes $ka = 2$ and $kb = 1$. The curves are computed with the T-matrix method and D-matrix method. Localized spherical vector wave functions are used for representing the solution.

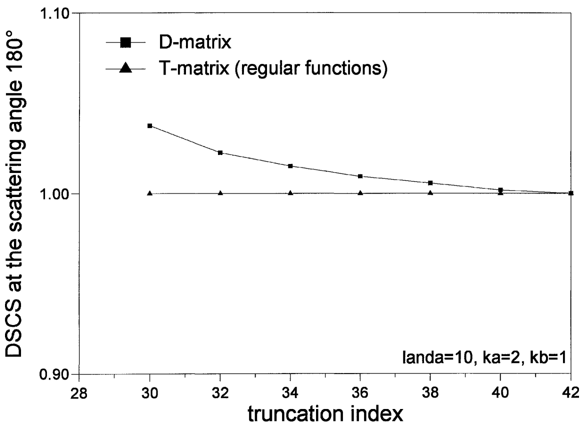

Fig. 2. The same as in Fig. 1 but the curves correspond to $\lambda = 10$.

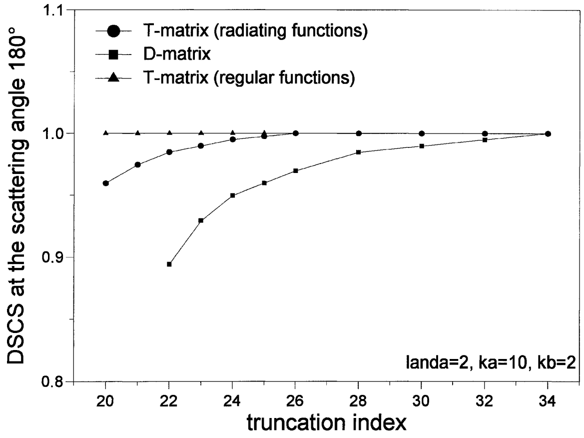

Fig. 3. Normalized differential scattering cross section (DSCS) at the scattering angle $\theta = 180°$ for different values of the truncation index. The scatterer is a prolate spheroid with $\lambda = 2$ and semiaxes $ka = 10$ and $kb = 2$. The curves are computed with the T-matrix method with regular and radiating functions for surface current densities approximation and the D-matrix method. Distributed spherical vector wave functions are used for representing the solution.

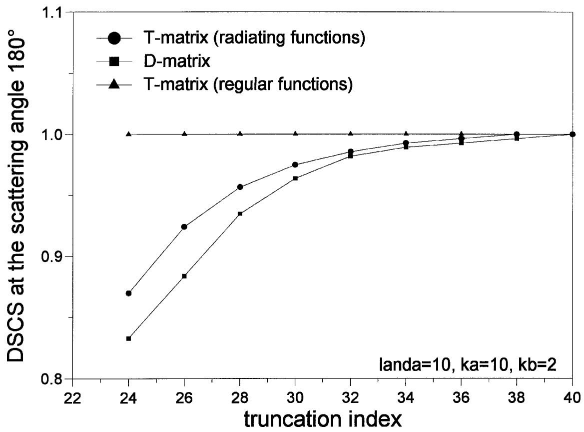

Fig. 4. The same as in Fig. 3 but the curves correspond to $\lambda = 10$.

matrix method and the D-matrix method. In order to achieve convergence we use (17) discrete sources in the first case, while 40 discrete sources are necessary in the second case. The discrepancy concerning the number of sources used is more pronounced if the aspect ratio increases. The results plotted in Fig. 6 correspond to a prolate spheroid with $ka = 10$ and $kb = 1$. As before, (17) discrete sources leads to accurate results in the case of the conventional method,

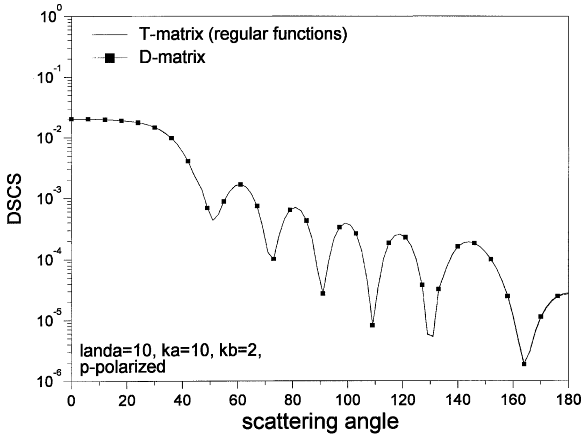

Fig. 5. Normalized differential scattering cross section (DSCS) for a prolate spheroid with $\lambda = 10$ and semiaxes $ka = 10$ and $kb = 2$. The curves are computed with the T-matrix method and D-matrix method. The basis and testing functions are the distributed spherical vector wave functions.

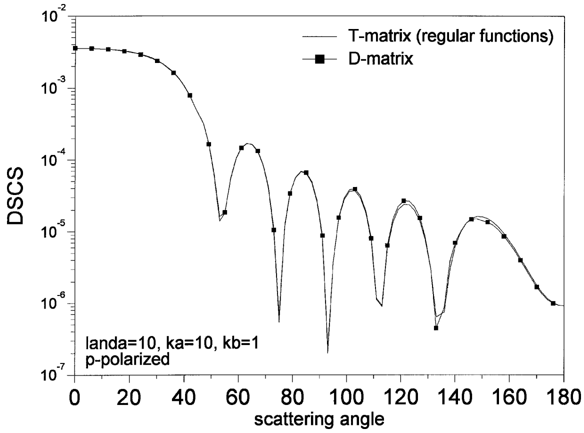

Fig. 6. Normalized differential scattering cross section (DSCS) for a prolate spheroid with $\lambda = 10$ and semiaxes $ka = 10$ and $kb = 1$. The curves are computed with the T-matrix method and D-matrix methods. The basis and testing functions are the distributed spherical vector wave functions.

while 50 discrete sources are required in the D-matrix method.

5. Conclusions

In this contribution the T-matrix method and a new projection scheme, called the D-matrix method, are investigated. In the case of the D-matrix method we proved the convergence and unique solvability of the linear system of equations when multipoles fields are used for surface current approximation. The numerical experiments indicate that the T-matrix method is the most efficient in spite of the fact that up to now there is no mathematical prove of its convergence. In this context the question of convergence of the conventional T-matrix method is open and several addition research is needed. Our contribution demonstrate once again the genial feeling of Waterman who chose simple derivation, preferring physical plausibility over mathematical rigor.

Acknowledgments

The support of Volkswagen-Foundation is gratefully acknowledged.

Appendix A

In this appendix we prove the completeness of the system

$$\left\{\mathbf{n}\times\mathbf{M}_\nu^3 + j\lambda\mathbf{n}\times(\mathbf{n}\times\mathbf{N}_\nu^3),\ \mathbf{n}\times\mathbf{N}_\nu^3 + j\lambda\mathbf{n}\times(\mathbf{n}\times\mathbf{M}_\nu^3)\right\}_{\nu=1}^{\infty}, \quad \operatorname{Re}\lambda \geqslant 0,$$

in $\mathcal{L}^2_{\tan}(S)$. Consider the vector potential with square integrable tangential density $\mathbf{a}$

$$\mathbf{A}_a(\mathbf{r}) = \int_S \mathbf{a}(\mathbf{r}')g(\mathbf{r},\mathbf{r}',k)\,dS', \quad \mathbf{r}\in\mathbf{R}^3 - S. \tag{19}$$

Then, the following lemma is valid.

Lemma 1. *Let*

$$\nabla\times\mathbf{A}_a(\mathbf{r}) - \lambda\frac{j}{k}\nabla\times\nabla\times\mathbf{A}_{n\times a}(\mathbf{r}) = 0, \quad \mathbf{r}\in D, \tag{20}$$

where $\mathbf{a}\in\mathcal{L}^2_{\tan}(S)$, $\operatorname{Im}k\geqslant 0$, *and* $\operatorname{Re}\lambda\geqslant 0$. *Then* $\mathbf{a}=0$ *almost everywhere on* S.

Proof. Define the electromagnetic field

$$\mathcal{E} = \nabla\times\mathbf{A}_a - \lambda\frac{j}{k}\nabla\times\nabla\times\mathbf{A}_{n\times a},$$
$$\mathcal{H} = \frac{1}{jk}\nabla\times\mathcal{E}. \tag{21}$$

From the hypothesis we get $\mathcal{E}=\mathcal{H}=0$ in D. From the jump relations for the vector potential with square integrable densities [7], we find that

$$\lim_{h\to 0_+}\left\|[\mathbf{n}\times\mathcal{E}(.+h\mathbf{n}(.))]-\mathbf{a}\right\|_{2,S}=0,$$
$$\lim_{h\to 0_+}\left\|[\mathbf{n}\times\mathcal{H}(.+h\mathbf{n}(.))]+\lambda\mathbf{n}\times\mathbf{a}\right\|_{2,S}=0. \tag{22}$$

For parallel exterior surfaces S_h, $S_h=\{\mathbf{r}=\mathbf{r}'+h\mathbf{n}(\mathbf{r}'),\ \mathbf{r}'\in S\}$, we make use of the following result [8]

$$-\lambda^*\int_S|\mathbf{n}\times\mathbf{a}|^2\,dS'$$
$$= -\lambda^*\lim_{h\to 0_+}\int_S\left[\mathbf{n}\times(\mathbf{n}\times\mathcal{E}(.+h\mathbf{n}(.)))\right] \times (\mathbf{n}\times\mathbf{a}^*)(1-2hH+h^2K)\,dS'$$
$$= \lim_{h\to 0_+}\int_{S_h}(\mathbf{n}\times\mathcal{E})\cdot\mathcal{H}^*\,dS', \tag{23}$$

where H and K represent the mean curvature and the Gaussian curvature of the surface, respectively. We consider now a spherical surface S_R of radius R enclosing D. Simple calculations show that $\mathcal{E}$ and $\mathcal{H}$ satisfies the weak form of the radiation condition, that is

$$\lim_{R\to\infty}\int_{S_R}|\mathcal{H}\times\mathbf{n}-\mathcal{E}|^2\,dS'$$
$$= \lim_{R\to\infty}\int_{S_R}\left\{|\mathcal{H}\times\mathbf{n}|^2+|\mathcal{E}|^2 - 2\operatorname{Re}\left[(\mathbf{n}\times\mathcal{E})\cdot\mathcal{H}^*\right]\right\}dS'=0. \tag{24}$$

Application of Gauss' divergence theorem in the region D_{hR}, bounded by the surface S_h and the spherical surface S_R gives

$$j\int_{D_{hR}}\left[k|\mathcal{H}|^2-k^*|\mathcal{E}|^2\right]dV'$$
$$= \int_{S_R}(\mathbf{n}\times\mathcal{E})\cdot\mathcal{H}^*\,dS' - \int_{S_h}(\mathbf{n}\times\mathcal{E})\cdot\mathcal{H}^*\,dS'. \tag{25}$$

Take the real part of the above equation, let $h\to 0_+$ and use (23). We get

$$-\operatorname{Re}(\lambda)\int_S |\mathbf{n}\times\mathbf{a}|^2\,dS'$$
$$=\operatorname{Re}\left\{\int_{S_R}(\mathbf{n}\times\mathcal{E})\cdot\mathcal{H}^*\,dS'\right\}$$
$$+\operatorname{Im}(k)\int_{D_R}\left[|\mathcal{H}|^2+|\mathcal{E}|^2\right]dV', \tag{26}$$

where

$$\lim_{h\to 0_+}\int_{D_{hR}}\left[|\mathcal{H}|^2+|\mathcal{E}|^2\right]dV'=\int_{D_R}\left[|\mathcal{H}|^2+|\mathcal{E}|^2\right]dV' \tag{27}$$

for any fixed R. Then,

$$-2\operatorname{Re}(\lambda)\int_S |\mathbf{n}\times\mathbf{a}|^2\,dS'$$
$$=\lim_{R\to\infty}\left\{\int_{S_R}\left(|\mathcal{H}\times\mathbf{n}|^2+|\mathcal{E}|^2\right)dS'\right.$$
$$\left.+\,2\operatorname{Im}(k)\int_{D_R}\left[|\mathcal{H}|^2+|\mathcal{E}|^2\right]dV'\right\}. \tag{28}$$

If $\operatorname{Re}(\lambda)>0$ the conclusion $\mathbf{a}\sim 0$ on S follows immediately. If $\operatorname{Re}(\lambda)=0$ and $\operatorname{Im}(k)>0$ we get $\mathcal{E}=0$ in $R^3-\overline{D}$. Finally, if $\operatorname{Re}(\lambda)=0$ and $\operatorname{Im}(k)=0$ we get $\int_S|\mathcal{E}|^2\,dS'\to 0$ as $R\to\infty$, whence $\mathcal{E}=0$ in $R^3-\overline{D}$ follows. Application of the jump relations (22) gives $\mathbf{a}\sim 0$ on S. □

Let us consider an approximation for the impedance boundary-value problem in the form of a linear combination of multipoles with a single origin, i.e.

$$\mathbf{E}_{sN}=\sum_{\mu=1}^{N}a_\mu^N\mathbf{M}_\mu^3+b_\mu^N\mathbf{N}_\mu^3,$$
$$\mathbf{H}_{sN}=-j\sum_{\mu=1}^{N}a_\mu^N\mathbf{N}_\mu^3+b_\mu^N\mathbf{M}_\mu^3. \tag{29}$$

Then the following theorem holds

Theorem 2. *For an arbitrary* $\mathbf{a}\in\mathcal{L}^2_{\tan}(S)$ *and any* $\delta\geqslant 0$ *, there exists* N *and a sequence* $\{a_\mu^N,b_\mu^N\}_{\mu=1}^N$ *such that*

$$\left\|\mathbf{n}\times\mathbf{E}_{sN}-\lambda\mathbf{n}\times(\mathbf{n}\times\mathbf{H}_{sN})-\mathbf{a}\right\|_{2,S}<\delta. \tag{30}$$

Proof. Since completeness and closeness imply each other, it is sufficient to prove that for any $\mathbf{a}\in\mathcal{L}^2_{\tan}(S)$

$$\int_S \mathbf{a}^*\cdot\begin{pmatrix}\mathbf{n}\times\mathbf{M}_\mu^3+j\lambda\mathbf{n}\times(\mathbf{n}\times\mathbf{N}_\mu^3)\\ \mathbf{n}\times\mathbf{N}_\mu^3+j\lambda\mathbf{n}\times(\mathbf{n}\times\mathbf{M}_\mu^3)\end{pmatrix}dS'=0,$$

$\mu=(m,n)=1,2,\cdots$ we obtain $\mathbf{a}=0$ in $\mathcal{L}^2_{\tan}(S)$. By successive transformations the above equation can be written as

$$\int_S\left[\mathbf{a}'\cdot\begin{pmatrix}\mathbf{M}_\mu^3\\ \mathbf{N}_\mu^3\end{pmatrix}-j\lambda(\mathbf{n}\times\mathbf{a}')\cdot\begin{pmatrix}\mathbf{N}_\mu^3\\ \mathbf{M}_\mu^3\end{pmatrix}\right]dS'=0, \tag{31}$$

where $\mathbf{a}'=\mathbf{n}\times\mathbf{a}^*$. Now, using using the spherical vector waves expansion of $\overline{\mathbf{I}}g$, where $\overline{\mathbf{I}}$ is the unit dyadic, we find that

$$\mathcal{E}=\nabla\times\mathbf{A}_{a'}-\lambda\frac{j}{k}\nabla\times\nabla\times\mathbf{A}_{n\times a'}$$

vanishes in D. Application of the above lemma gives $\mathbf{a}\sim 0$ on S and the theorem is proved. □

Appendix B

In this appendix we analyze the convergence of the D-matrix method. To this end let us shown the following theorems to be valid.

Theorem 3. *System* (15) *is unique solvable for any fixed* N.

Proof. We establish the dissipativity of the **D**-matrix, i.e. we establish that the inequality $\operatorname{Im}\langle\mathbf{DT}_N,\mathbf{T}_N\rangle_{l^2}<0$ holds for any $\mathbf{T}_N=[a_\mu^N,b_\mu^N]_{\mu=1}^N$, and in addition from $\langle\mathbf{DT}_N,\mathbf{T}_N\rangle_{l^2}=0$ we receive $\mathbf{T}_N=0$. We choose an arbitrary $\mathbf{T}_N$ and construct the vector fields $\mathbf{E}_{sN}$ and $\mathbf{H}_{sN}$ according to (13). It is readily seen that

$$\langle\mathbf{DT}_N,\mathbf{T}_N\rangle_{l^2}$$
$$=-j\left\langle\mathbf{n}\times\mathbf{E}_{sN}-\lambda\mathbf{n}\times(\mathbf{n}\times\mathbf{H}_{sN}),\mathbf{H}_{sN}\right\rangle_{2,S}. \tag{32}$$

Taking the imaginary part of this equation and using

$$\operatorname{Re}\left\{\int_S(\mathbf{n}\times\mathbf{E}_{sN})\cdot\mathbf{H}_{sN}^*\,dS'\right\}=\int_\Omega\left|\mathbf{E}_{s0}^N\right|^2d\omega \tag{33}$$

we obtain

$$\operatorname{Im}\langle \mathbf{D}\mathbf{T}_N, \mathbf{T}_N\rangle_{l^2} = -\int_\Omega |\mathbf{E}_{sN0}|^2 d\omega - \int_S \operatorname{Re}\lambda |\mathbf{n}\times\mathbf{H}_{sN}|^2 dS'. \quad (34)$$

Since $\operatorname{Re}\lambda > 0$, (34) shows the dissipativity of the matrix $\mathbf{D}$. Now, from $\langle \mathbf{D}\mathbf{T}_N, \mathbf{T}_N\rangle_{l^2} = 0$ we receive $\mathbf{E}_{sN0} = 0$, whence by the orthogonality of the spherical harmonics on the unit sphere, $\mathbf{T}_N = 0$ follows. □

Theorem 4. *Let the approximate solution to the impedance boundary-value problem be given by* (13), *where the amplitude vector* $\mathbf{T}_N = [a_\mu^N, b_\mu^N]_{\mu=1}^N$ *solves the system* (15). *Then*

$$\lim_{N\to\infty} \|\mathbf{E}_{s0} - \mathbf{E}_{sN0}\|_{2,\Omega} = 0, \quad (35)$$

where $\mathbf{E}_{s0}$ *is the far-field pattern of the exact solution* $\mathbf{E}_s, \mathbf{H}_s$.

Proof. Let us shown that the sequence $\mathbf{H}_{sN}$ is bounded for any N. In (14) we multiply the first set of equations by ja_ν^*, the second one by jb_ν^* and sum the resulting expressions. We get as before

$$\langle \mathbf{n}\times\mathbf{E}_{sN} - \lambda\mathbf{n}\times(\mathbf{n}\times\mathbf{H}_{sN}) - \mathbf{f}, \mathbf{H}_{sN}\rangle_{2,S} = 0. \quad (36)$$

Taking the real part of (36), using (33), and the identity $|\mathbf{a}-\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\operatorname{Re}(\mathbf{a}\cdot\mathbf{b}^*)$, we find that

$$\int_S \frac{1}{4\operatorname{Re}\lambda}|\mathbf{f}|^2 dS' = \int_\Omega |\mathbf{E}_{sN0}|^2 d\omega + \int_S \left|\sqrt{\operatorname{Re}\lambda}\,\mathbf{n}\times\mathbf{H}_{sN} - \frac{1}{2\sqrt{\operatorname{Re}\lambda}}\mathbf{n}\times\mathbf{f}\right|^2 dS'. \quad (37)$$

From this relation it follows directly that the sequence $\mathbf{H}_{sN}$ is bounded for any N.

The completeness of the system

$$\{\mathbf{n}\times\mathbf{\Psi}_\nu^3 + j\lambda\mathbf{n}\times(\mathbf{n}\times\mathbf{\Phi}_\nu^3),\ \mathbf{n}\times\mathbf{\Phi}_\nu^3 + j\lambda\mathbf{n}\times(\mathbf{n}\times\mathbf{\Psi}_\nu^3)\}_{\nu=1}^\infty$$

yields the existence of vector fields

$$\mathcal{E}_{sN} = \sum_{\mu=1}^N \alpha_\mu^N \mathbf{\Psi}_\mu^3 + \beta_\mu^N \mathbf{\Phi}_\mu^3,$$

$$\mathcal{H}_{sN} = -j\sum_{\mu=1}^N \alpha_\mu^N \mathbf{\Phi}_\mu^3 + \beta_\mu^N \mathbf{\Psi}_\mu^3, \quad (38)$$

such that

$$\lim_{N\to\infty} \|\boldsymbol{\varepsilon}_N\|_{2,S} = 0 \quad (39)$$

and

$$\lim_{N\to\infty} \|\mathbf{E}_{s0} - \mathcal{E}_{sN0}\|_{2,\Omega} = 0. \quad (40)$$

Here, $\boldsymbol{\varepsilon}_N = \mathbf{n}\times\mathcal{E}_{sN} - \lambda\mathbf{n}\times(\mathbf{n}\times\mathcal{H}_{sN}) - \mathbf{f}$ represents the discrepancies of the tangential fields on the surface S, and $\mathcal{E}_{sN0}$ is the far-field pattern of the approximate solution $\mathcal{E}_{sN}, \mathcal{H}_{sN}$. Then, we use (14) and

$$\langle \mathbf{n}\times\mathcal{E}_{sN} - \lambda\mathbf{n}\times(\mathbf{n}\times\mathcal{H}_{sN}) - \mathbf{f}, \mathbf{\Phi}_\nu^3\rangle_{2,S} = \langle \boldsymbol{\varepsilon}_N, \mathbf{\Phi}_\nu^3\rangle_{2,S},$$
$$\langle \mathbf{n}\times\mathcal{E}_{sN} - \lambda\mathbf{n}\times(\mathbf{n}\times\mathcal{H}_{sN}) - \mathbf{f}, \mathbf{\Psi}_\nu^3\rangle_{2,S} = \langle \boldsymbol{\varepsilon}_N, \mathbf{\Psi}_\nu^3\rangle_{2,S} \quad (41)$$

to derive the following set of equations

$$\langle \mathbf{n}\times\delta\mathbf{E}_{sN} - \lambda\mathbf{n}\times(\mathbf{n}\times\delta\mathbf{H}_{sN}), \mathbf{\Phi}_\nu^3\rangle_{2,S} = \langle \boldsymbol{\varepsilon}_N, \mathbf{\Phi}_\nu^3\rangle_{2,S},$$
$$\langle \mathbf{n}\times\delta\mathbf{E}_{sN} - \lambda\mathbf{n}\times(\mathbf{n}\times\delta\mathbf{H}_{sN}), \mathbf{\Psi}_\nu^3\rangle_{2,S} = \langle \boldsymbol{\varepsilon}_N, \mathbf{\Psi}_\nu^3\rangle_{2,S} \quad (42)$$

for the residual fields $\delta\mathbf{E}_{sN} = \mathcal{E}_{sN} - \mathbf{E}_{sN}$ and $\delta\mathbf{H}_{sN} = \mathcal{H}_{sN} - \mathbf{H}_{sN}$. Since $\delta\mathbf{E}_{sN}$ and $\delta\mathbf{H}_{sN}$ are expressed as linear combinations of $\mathbf{\Psi}_\mu^3$ and $\mathbf{\Phi}_\mu^3$, with $1 \leqslant \mu \leqslant N$, we conclude that

$$\int_S \left[(\mathbf{n}\times\delta\mathbf{E}_{sN})\cdot\delta\mathbf{H}_{sN}^* + \lambda|\mathbf{n}\times\delta\mathbf{H}_{sN}|^2\right] dS' = \int_S \boldsymbol{\varepsilon}_N\cdot\delta\mathbf{H}_{sN}^* dS' \quad (43)$$

or, equivalently that

$$\int_\Omega |\delta\mathbf{E}_{sN0}|^2 d\omega + \int_S \operatorname{Re}\lambda|\mathbf{n}\times\delta\mathbf{H}_{sN}|^2 dS' = \operatorname{Re}\left\{\int_S \varepsilon_N\cdot\delta\mathbf{H}_{sN}^* dS'\right\} \leqslant \|\boldsymbol{\varepsilon}_N\|_{2,S}\|\delta\mathbf{H}_{sN}\|_{2,S}. \quad (44)$$

The uniform boundedness of the sequence $\mathbf{H}_{sN}$ with respect to N and (44) gives

$$\lim_{N\to\infty} \|\mathbf{E}_{sN0} - \mathcal{E}_{sN0}\|_{2,\Omega} = 0$$

and finally, according to (40) we conclude that (35) holds. □

An inspection of (16) reveals that in the case of axisymmetric scatterers the surface integrals simplify to one-dimensional integrals along the particle generatrix. The problem decouples over the azimuthal modes and therefore the amount of computer storage required to solve the scattering problem is not excessive high. In contrast, for particles without rotational symmetry it is not possible to obtain a separate solution for each azimuthal mode. Consequently, the dimensions of the linear systems of equations considerably increases. This leads to increased difficulties that are associated with the stability of the solutions. Effective solutions of these systems appear to be possible only by means of iterative schemes. It is therefore reasonable to analyze the correct solvability of system (15) in order to investigate the applicability of iterative schemes. For the system of localized spherical vector wave functions we can prove the following theorem

Theorem 5. *System* (15) *is correctly solvable for any fixed* N.

Proof. We have

$$\begin{aligned}\left|\langle \mathbf{D}\mathbf{T}_N, \mathbf{T}_N\rangle_{l^2}\right| &\geqslant -\operatorname{Im}\langle \mathbf{D}\mathbf{T}_N, \mathbf{T}_N\rangle_{l^2}\\ &= \int_{\Omega} |\mathbf{E}_{sN0}|^2 d\omega + \int_{S} \operatorname{Re}\lambda |\mathbf{n}\times\mathbf{H}_{sN}|^2 dS'\\ &\geqslant \int_{\Omega} |\mathbf{E}_{sN0}|^2 d\omega = \sum_{\mu=1}^{N} \frac{1}{k^2}\frac{\pi}{D_\mu}\left(|a_\mu^N|^2 + |b_\mu^N|^2\right)\\ &\geqslant C\|\mathbf{T}_N\|_{l^2}^2. \qquad (45)\end{aligned}$$

This implies the correct solvability of the system (15), i.e.

$$\|\mathbf{D}\mathbf{T}_N\|_{l^2} \geqslant C\|\mathbf{T}_N\|_{l^2}. \qquad \square$$

In this context it is possible to apply iterative schemes such as, e.g., GMRES [9] for any fixed value of the truncation index N.

References

[1] P.C. Waterman, New formulation of acoustic scattering, J. Acoust. Soc. Am. 45 (1969) 1417–1429.
[2] A.G. Ramm, Convergence of the T-matrix approach to scattering theory, J. Math. Phys. 23 (1982) 1123–1125.
[3] G. Kristensson, A.G. Ramm, S. Ström, Convergence of the T-matrix approach to scattering theory. II, J. Math. Phys. 24 (1983) 2619–2631.
[4] T. Wriedt, A. Doicu, Formulations of the extended boundary condition method for three-dimensional scattering using the method of discrete sources, J. Modern Optics 45 (1998) 199–213.
[5] Yu.A. Eremin, A.G. Sveshnikov, A projective-iterative scheme for determining the amplitudes of discrete sources on the basis of dissipative matrices, Comput. Math. and Math. Physics 37 (1997) 219–226.
[6] D. Colton, R. Kress, The impedance boundary value problem for the time harmonic Maxwell equations, Math. Meth. in Appl. Sci. 3 (1981) 475–487.
[7] D. Colton, R. Kress, Dense sets and far field patterns in electromagnetic wave propagation, SIAM J. Math. Anal. 16 (1985) 1049–1061.
[8] P. Hähner, An exterior boundary-value problem for the Maxwell equations with boundary data in Sobolev space, Proc. Royal Soc. Edinburgh 109 (1988) 213–224.
[9] Y. Saad, M.H. Schulz, GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems, SIAM J. Sci. Stat. Comput. 7 (1986) 856–869.