Computation of the beam-shape coefficients in the generalized Lorenz–Mie theory by using the translational addition theorem for spherical vector wave functions

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The generalized Lorenz–Mie theory describes the electromagnetic scattering of a Gaussian laser beam by a spherical particle. The most intensive computational aspect of the theory concerns the evaluation of the beam-shape coefficients in the general case of an off-axis location of the scatterer. These beam-shape coefficients can be computed starting from the set of beam-shape coefficients for an on-axis location by using the addition theorem for the spherical vector wave functions of the first kind under a translation of the coordinate origin.

1. Introduction

The existence of a generalization of the Lorenz-Mie theory (GLMT) to the case in which a spherical, homogeneous, isotropic particle is illuminated by a Gaussian beam is of importance in various fields of optics, such as the design of optical sizing systems and the interpretation of signals obtained from them. There are a number of mathematical theories for arbitrary beam scattering. Each of these theories relies on the decomposition of the incident beam into an infinite series of elementary constituents, such as partial waves[1–5] or plane waves,[6] with amplitudes and phases given by a set of beam-shape coefficients. The beam-shape coefficients may be evaluated by using quadratures,[4,5,7,8] a finite series technique,[9] or localized approximations.[7,8,10,11] Because quadrature techniques have to integrate radial components of the incident field that are rapidly varying functions, they are very time consuming; conversely, the use of localized approximations is the most efficient technique.

From a mathematical point of view, one fundamental problem in the GLMT concerns the expansion of the incident Gaussian beam as a series of spherical vector wave functions (SVWF's). The GLMT description of the beam may be based on Davis's approximations,[12] which account for axial and transverse field components. Each nth Davis beam appears under three versions, which are (a) the mathematical conservative version, (b) the L version, and (c) the symmetrized version.[7] None of these beams are exact solutions of Maxwell's equations. Each nth Davis beam can be considered as a pseudoelectromagnetic field. It is known that when the electromagnetic description of the Gaussian beams does not perfectly satisfy Maxwell's equations, the expansion in terms of the SVWF is problematic and both mentioned quadratures methods are basically flawed. These flaws do not, however, prevent an accurate evaluation of beam-shape coefficients when their nature is correctly identified.

For an on-axis location of the scatterer, the incident beam may be described by a set of g_n beam-shape coefficients instead of by a double set $g_{n,\mathrm{TM}}{}^m$, $g_{n,\mathrm{TE}}{}^m$ in the off-axis case.[7,13] Once these coefficients are determined in an on-axis coordinate system, the more general coefficients $g_n{}^m$ can be expressed by using an addition theorem for the SVWF of the first kind under a translation of the coordinate origin. The aim of this paper is to provide an exhaustive discussion of this procedure.

The body of this paper is organized as follows. In Section 2 we give a short summary of the equivalence methods for arbitrary incident beams. In Section 3 we derive the relations between the $g_n{}^m$ coefficients and the g_n coefficients by using an addition theorem

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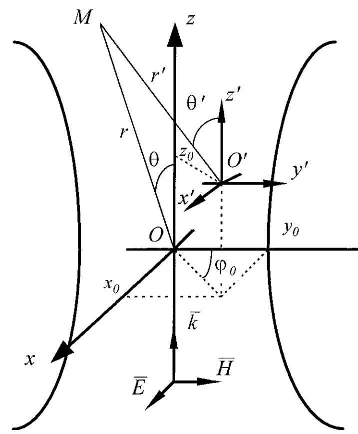

Fig. 1. Geometry of the problem: beam coordinate system $Oxyz$ and particle-location system $O'x'y'z'$. The Gaussian beam travels in the z direction and is polarized in the x direction.

for the SVWF under a translation of the coordinate origin. In Section 4 we give several integral representations for the beam-shape coefficients, and finally we provide a justification of the localized approximation to on-axis and off-axis Gaussian beams.

2. Equivalence Methods for Arbitrary Incident Beams

We consider an electromagnetic wave of wavelength λ and wave number k, and we define the original beam coordinate system $Oxyz$ so that the center of the Gaussian beam waist is located at point O. The Gaussian beam travels in the z direction and is polarized in the x direction. A particle is illuminated by the Gaussian beam. The center of the scatterer is located at the point O' of a Cartesian coordinate system $O'x'y'z'$, with Ox parallel to $O'x'$ and with similar conditions for the other axes. This is shown in Fig. 1. The set of spherical coordinates (r', θ', φ') is introduced with respect to O', while (r, θ, φ) represents the set of spherical coordinates in the beam coordinate system. The Cartesian coordinates of O' in the system $Oxyz$ are (x_0, y_0, z_0), and by convention we call this frame the particle-location system. We denote by $\rho_0 = \sqrt{{x_0}^2 + {y_0}^2}$, $\varphi_0 = \arctan(x_0/y_0)$, and z_0 the cylindrical coordinates of the particle center in the beam frame.

We start our presentation with a general analysis of arbitrary incident beams. Let S be a spherical surface with radius R and center O. We denote by $L_2(S)$ the space of the complex square integrable tangential vector functions on S, with the scalar product

$$\langle \bar{f}, \bar{g} \rangle = \int_S \bar{f} \cdot \bar{g}^* \mathrm{d}S, \tag{1}$$

where the asterisk represents the complex conjugate.

Let us consider a pseudoelectromagnetic field with electric component $\bar{E}(\bar{r})$. To obtain an expansion of the pseudoelectromagnetic field in terms of the SVWF, we must perform an equivalence between the original field and an electromagnetic field that satisfies the vector Helmholtz equation and is divergence free. One possible way to perform such an equivalence is to introduce an equivalent electromagnetic field $\bar{E}^R(\bar{r})$, satisfying the vector Helmholtz equation inside S, the boundary condition

$$\bar{n} \times \bar{E}(\bar{r}') = \bar{n} \times \bar{E}^R(\bar{r}') = \bar{e}(\bar{r}'), \quad \bar{r}' \in S, \bar{e} \in L_2(S), \tag{2}$$

and having zero divergence. Here $\bar{n}$ is the normal vector exterior to S.

If k^2 is not an eigenvalue corresponding to resonance frequencies of the interior region, then the tangential components of the SVWF of the first kind $\{\bar{n} \times \bar{M}_{mn}{}^1(k\bar{r}'),\ \bar{n} \times \bar{N}_{mn}{}^1(k\bar{r}')\}_{n=1,2,\ldots;m=\overline{-n,n}}$ form a complete, orthogonal set of functions on $L_2(S)$. In this case for any $\delta > 0$, there exists $N(\delta)$ and

$$\bar{e}_N(\bar{r}') = \sum_{n=1}^{N} \sum_{m=-n}^{n} C_{nm}\{ig_{n,\mathrm{TE}}{}^m(R)[\bar{n} \times \bar{M}_{mn}{}^1(\bar{k}\bar{r}')] + g_{n,\mathrm{TM}}{}^m(R)[\bar{n} \times \bar{N}_{mn}{}^1(\bar{k}\bar{r}')]\}, \tag{3}$$

such that

$$\|\bar{e} - \bar{e}_N\| \leq \delta. \tag{4}$$

Here C_{nm} is a normalization constant and

$$ig_{n,\mathrm{TE}}{}^m(R) = \frac{1}{E_0} \langle \bar{e}, \bar{n} \times \bar{M}_{mn}{}^1 \rangle,$$

$$g_{n,\mathrm{TM}}{}^m(R) = \frac{1}{E_0} \langle \bar{e}, \bar{n} \times \bar{N}_{mn}{}^1 \rangle \tag{5}$$

are expansion coefficients depending on R. By this procedure we have constructed an approximation of the equivalent electromagnetic field in the form

$$\bar{E}_N{}^R(\bar{r}) = E_0 \sum_{n=1}^{N} \sum_{m=-n}^{n} C_{nm}[ig_{n,\mathrm{TE}}{}^m(R)\bar{M}_{mn}{}^1(k\bar{r}) + g_{n,\mathrm{TM}}{}^m(R)\bar{N}_{mn}{}^1(k\bar{r})], \tag{6}$$

where E_0 is the electric field strength.

We note that $\bar{E}^R(\bar{r})$ depends on the radius of the spherical surface where the boundary condition was imposed. This formulation is analogous with the plane-wave spectrum method, which ensures the equality of the pseudoelectromagnetic field with an equivalent electromagnetic field in a given plane.

Suppose now that we are able to separate the terms of the $g_n{}^m(R)$ coefficients that are independent from R:

$$g_{n,X}{}^m(R) = g_{n,X}{}^m + \delta g_{n,X}{}^m(R), \quad X = \mathrm{TE}, \mathrm{TM}. \tag{7}$$

If the sequence

$$\bar{e}_N{}^0(\bar{r}') = \sum_{n=1}^{N} \sum_{m=-n}^{n} C_{nm}\{ig_{n,\mathrm{TE}}{}^m[\bar{n} \times \bar{M}_{mn}{}^1(\bar{k}\bar{r}')] + g_{n,\mathrm{TM}}{}^m[\bar{n} \times \bar{N}_{mn}{}^1(\bar{k}\bar{r}')]\} \tag{8}$$

is convergent in the norm, we have obtained an approximation

$$\bar{E}_N{}^0(\bar{r}) = E_0 \sum_{n=1}^{N} \sum_{m=-n}^{n} C_{nm}[ig_{n,\mathrm{TE}}{}^m \bar{M}_{mn}{}^1(k\bar{r}) + g_{n,\mathrm{TM}}{}^m \bar{N}_{mn}{}^1(k\bar{r})] \tag{9}$$

of what we have called the electromagnetic component of the incident field $\bar{E}^0(\bar{r})$.

It is clear that $\bar{E}_N{}^R(\bar{r})$ is different from $\bar{E}_N{}^0(\bar{r})$. If $\bar{E}(\bar{r})$ is an electromagnetic field that satisfies Maxwell's equation, then $\bar{E}_N{}^R(\bar{r}) = \bar{E}_N{}^0(\bar{r})$ and both series converge uniformly to $\bar{E}(\bar{r})$. The electromagnetic component of the incident field is independent of the radius of the spherical surface, so if we have obtained a series representation in a given coordinate system, a new expansion can be derived in any translated reference system by using a translational addition theorem for the SVWF. The main problem with this formulation consists in the computation of the R-independent terms of the beam-shape coefficients. Such a separation was performed by Lock and Gouesbet for an on-axis fifth-order Barton symmetrized beam and for an off-axis first-order Barton symmetrized field.[7,8]

3. Relations between the $g_n{}^m$ Coefficients and the g_n Coefficients

We assume an expansion of the electromagnetic component of the incident field in terms of the SVWF, i.e.,

$$\bar{E}^0(\bar{r}) = E_0 \sum_{n\geq 1} \sum_{m=-n}^{n} C_{nm}[ig_{n,\mathrm{TE}}{}^m \bar{M}_{mn}{}^1(k\bar{r}) + g_{n,\mathrm{TM}}{}^m N_{mn}{}^1(k\bar{r})]. \tag{10}$$

The Legendre functions that appear in the expressions of the SVWF in Eq. (10) are defined for negative and positive values of index m. In this case, the C_{nm} coefficients are normalization factors for negative values of the index m introduced to ensure the compatibility between our formalism and the classical one[1–4]:

$$C_{nm} = \begin{cases} C_n, & m \geq 0 \\ (-1)^{|m|} \dfrac{(n+|m|)!}{(n-|m|)!} C_n, & m < 0. \end{cases} \tag{11}$$

The coefficients C_n appear in the Bromwich formulation of the Lorenz–Mie theory and are given by[1–4]

$$C_n = i^{n-1} \frac{2n+1}{n(n+1)}. \tag{12}$$

When the particle is located at the beam waist center, only the coefficients that correspond to the azimuthal modes $m = \pm 1$ are required to describe the incident field. The symmetrized version is superior to the other two Davis beam versions because it leads to a GLMT description of the beam in terms of one single set of coefficients g_n, whatever the considered Davis order, whereas the mathematical conservative and the L versions require the use of two sets $g_{n,\mathrm{TM}}$ and $g_{n,\mathrm{TE}}$ associated with the electric and magnetic fields.[7] In this special case the GLMT dramatically simplifies, and the double set reduces to a single set of coefficients g_n:

$$g_{n,\mathrm{TM}}{}^1(R) = g_{n,\mathrm{TM}}{}^{-1}(R) = \tfrac{1}{2} g_n(R),$$

$$g_{n,\mathrm{TE}}{}^1(R) = -g_{n,\mathrm{TE}}{}^{-1}(R) = -\tfrac{i}{2} g_n(R). \tag{13}$$

On-axis beam-shape coefficients for a weakly focused Gaussian beam are practically independent of the radial evaluation point. However, for a tightly focused Gaussian beam with $s = 0.1$, where s is a fundamental parameter equal to the waist radius w_0 divided by the diffraction length l, the influence of R is significant. This influence is, however, minimal when the symmetrized Barton version of the Davis beam is used to evaluate beam-shape coefficients. It was demonstrated that the g_n coefficients [the independent part of $g_n(R)$ with respect to R] computed by using the localized approximation are very closely related to the Barton symmetrized fifth-order beam-shape coefficients.[7] In this case one can assume that the coefficients g_n can be computed without any loss of accuracy in most relevant situations as

$$g_n = \exp[-s^2(n+0.5)^2]. \tag{14}$$

The expression of the electromagnetic component of the incident field written in the beam coordinate system by using the coefficients g_n's can be generalized in the particle-location system by using a translational addition theorem for the SVWF:

$$\begin{bmatrix} \bar{M}_{mn}{}^1(k\bar{r}) \\ N_{mn}{}^1(k\bar{r}) \end{bmatrix} = \sum_{n'\geq 1} \sum_{m'=-n'}^{n'} \begin{bmatrix} A_{m'n'}{}^{mn} \\ B_{m'n'}{}^{mn} \end{bmatrix} \bar{M}_{m'n'}{}^1(k\bar{r}') + \begin{bmatrix} B_{m'n'}{}^{mn} \\ A_{m'n'}{}^{mn} \end{bmatrix} \bar{N}_{m'n'}{}^1(k\bar{r}'). \tag{15}$$

This procedure leads to a relation between the complete set of beam-shape coefficients and the g_n's. The beam-shape coefficients are then expressed as a series of the g_n coefficients involving the translation coefficients

$$\begin{bmatrix} g_{n',\mathrm{TM}}{}^{m'} \\ ig_{n',\mathrm{TE}}{}^{m'} \end{bmatrix} = \frac{1}{2C_{n'm'}} \sum_{n\geq 1} [\pm C_{n,-1}(A_{m'n'}{}^{-1,n} - B_{m'n'}{}^{-1,n}) + C_{n1}(A_{m'n'}{}^{1,n} + B_{m'n'}{}^{1,n})] g_n. \tag{16}$$

The problem of computing the beam-shape coefficients gets simplified to a problem in which the translation coefficients $A_{m'n'}{}^{mn} B_{m'n'}{}^{mn}$ are required.

The beam-shape coefficients computed by using the addition theorem are compared with the off-axis localized approximation to these coefficients. The localized approximation to the Davis first-order beam-shape coefficients is an analytical function whose use greatly simplifies Gaussian beam-scattering calculation. It is[8,11,14]

$$\begin{bmatrix} g_{n,\mathrm{TM}}{}^{m,\mathrm{loc}} \\ ig_{n,\mathrm{TE}}{}^{m,\mathrm{loc}} \end{bmatrix} = (-1)^{m-1} K_{nm} \bar{\Psi}_0{}^0 \exp(ikz_0) \frac{1}{2} \times \left\{ \exp[i(m-1)\varphi_0] J_{m-1}\left(2\frac{\bar{Q}\rho_0\rho_n}{w_0{}^2}\right) \pm \exp[i(m+1)\varphi_0] J_{m+1}\left(2\frac{\bar{Q}\rho_0\rho_n}{w_0{}^2}\right)\right\}, \tag{17}$$

where

$$\bar{\Psi}_0{}^0 = i\bar{Q}\exp(-i\bar{Q}\rho_0{}^2/w_0{}^2)\exp[-i\bar{Q}(n+0.5)^2/(k^2 w_0{}^2)],$$

$$K_{nm} = \begin{cases} (-i)^{|m|} \dfrac{i}{(n+0.5)^{|m|-1}}, & m \neq 0 \\ \dfrac{n(n+1)}{n+0.5}, & m = 0, \end{cases}$$

$$\rho_n = (n+0.5)\frac{1}{k}, \ \bar{Q} = 1/(i - 2z_0/l). \tag{18}$$

Equation (17) is the off-axis generalization of the localized approximations coefficients for an on-axis beam, where the coefficients are nonzero only for $m = \pm 1$.

4. Integral Representation for the $g_n{}^m$ Coefficients

Addition theorems for spherical vector wave functions, based on corresponding theorems for the spherical scalar wave functions, can be found in the original papers of Friedman and Russek,[15] Stein,[16] and Cruzan.[17] The translation coefficients are expressed in terms of the linearization expansion coefficients of the Legendre functions $a(*)$. The $a(*)$ coefficients can be identified with a product of two Wigner 3-j symbols, which are associated with the coupling of two angular momentum eigenvectors. The Wigner 3-j symbols involve summations of multitudes of factorials so that straightforward calculations of $a(*)$ that use the direct representation are inefficient. An efficient method to compute the translation coefficients requires a recursion relation for $a(*)$. Such a relation was given by Bruning and Lo[18] for the translation along the z axis. For translation in any other direction, one can use a recursion relation for the Clebsch–Gordan coefficients and the relation between the Clebsh–Gordan coefficients and the Wigner 3-j symbols[19,20] to extend Bruning's relation.

In our study we have used an addition theorem for the cylindrical wave functions of the first kind and a series expansion of the cylindrical wave vectors in terms of the spherical vector wave functions to obtain an integral representation for the translation coefficients

$$A_{m'n'}{}^{mn} = \frac{i^{n'-n}}{2} \frac{2n'+1}{n'(n'+1)} \frac{(n'-m')!}{(n'+m')!} \times \int_0^\pi J_{m'-m}(k\rho_0 \sin\alpha)\exp \times [i(m'-m)\varphi_0]\exp(ik\cos\alpha z_0) \times \left(\frac{\mathrm{d}P_{n'}{}^{m'}}{\mathrm{d}\alpha}\frac{\mathrm{d}P_n{}^m}{\mathrm{d}\alpha} + mm'\frac{P_{n'}{}^{m'}}{\sin\alpha}\frac{P_n{}^m}{\sin\alpha}\right)\sin\alpha \mathrm{d}\alpha,$$

$$B_{m'n'}{}^{mn} = \frac{i^{n'-n}}{2} \frac{2n'+1}{n'(n'+1)} \frac{(n'-m')!}{(n'+m')!} \times \int_0^\pi J_{m'-m}(k\rho_0 \sin\alpha)\exp \times [i(m'-m)\varphi_0]\exp(ik\cos\alpha z_0) \times \left(m\frac{\mathrm{d}P_{n'}{}^{m'}}{\mathrm{d}\alpha}\frac{P_n{}^m}{\sin\alpha} + m'\frac{P_{n'}{}^{m'}}{\sin\alpha}\frac{\mathrm{d}P_n{}^m}{\mathrm{d}\alpha}\right)\sin\alpha \mathrm{d}\alpha. \tag{19}$$

The argument of the Legendre functions in Eq. (19) is $\cos\alpha$.

By inserting Eq. (19) into Eq. (16) and reducing the integration interval to 0, $\pi/2$, we obtain

$$\begin{bmatrix} g_{n',\mathrm{TM}}{}^{m'} \\ ig_{n',\mathrm{TE}}{}^{m'} \end{bmatrix} = \frac{1}{4}\frac{(n'-|m'|)!}{(n'+|m'|)!} \times \left\{ \int_0^{\pi/2} \exp(ik\cos\alpha z_0) \times [\Psi_{n'm'}(\alpha) T_{m'-1}(\sin\alpha) \pm \Psi_{n',-m'}(\alpha) T_{m'+1}(\sin\alpha)] S_1(\alpha)\sin\alpha \mathrm{d}\alpha + (-1)^{n'+m'+1}\int_0^{\pi/2}\exp(-ik\cos\alpha z_0) \times [\Psi_{n',-m'}(\alpha) T_{m'-1}(\sin\alpha) \pm \Psi_{n',m'}(\alpha) T_{m'+1}(\sin\alpha)] S_2(\alpha)\sin\alpha \mathrm{d}\alpha \right\}, \tag{20}$$

where

$$T_{m'}(\alpha) = J_{m'}(k\rho_0 \sin\alpha)\exp(im'\varphi_0)$$

$$\Psi_{n'm'}(\alpha) = \frac{\mathrm{d}P_{n'}{}^{|m'|}(\cos\alpha)}{\mathrm{d}\alpha} + m'\frac{P_{n'}{}^{|m'|}(\cos\alpha)}{\sin\alpha}, \tag{21}$$

$$S_1(\alpha) = \sum_{n\geq 1} \frac{2n+1}{n(n+1)}\left(\frac{\mathrm{d}P_n{}^1}{\mathrm{d}\alpha} + \frac{P_n{}^1}{\sin\alpha}\right) \times \exp[-s^2(n+0.5)^2],$$

$$S_2(\alpha) = \sum_{n\geq 1} \frac{2n+1}{n(n+1)}(-1)^n\left(\frac{\mathrm{d}P_n{}^1}{\mathrm{d}\alpha} - \frac{P_n{}^1}{\sin\alpha}\right) \times \exp[-s^2(n+0.5)^2]. \tag{22}$$

Table 1. Comparison between $g_{n,\mathrm{TM}}{}^{m}$ Coefficients Computed in the Localized Approximation and Addition Theorem Methods

		Addition Theorem		
m n	Localized Approximation	Eq. (20)	Integral Representation, Eq. (26)	Simplified Integral Representation, Eq. (30)
0	9.82726×10^{-4}	9.82153×10^{-4}	9.84015×10^{-4}	9.82572×10^{-4}
1	9.16325×10^{-3}	9.16225×10^{-3}	9.16715×10^{-3}	9.15954×10^{-3}
0	2.94447×10^{-3}	2.94400×10^{-3}	2.94833×10^{-3}	2.94316×10^{-3}
2	2.74665×10^{-2}	2.74618×10^{-2}	2.74782×10^{-2}	2.74576×10^{-2}
0	5.87785×10^{-3}	5.87745×10^{-3}	5.88553×10^{-3}	5.87546×10^{-3}
3	5.48635×10^{-2}	5.48531×10^{-2}	5.48531×10^{-2}	5.48471×10^{-2}
0	9.77181×10^{-3}	9.77004×10^{-3}	9.78454×10^{-3}	9.76803×10^{-3}
4	9.12850×10^{-2}	9.12690×10^{-2}	9.13235×10^{-2}	9.12589×10^{-2}
0	1.46116×10^{-2}	1.46083×10^{-2}	1.46306×10^{-2}	1.46062×10^{-2}
5	1.36638×10^{-1}	1.36616×10^{-1}	1.36616×10^{-1}	1.36601×10^{-1}
1	3.62323×10^{-1}	3.62364×10^{-1}	3.62367×10^{-1}	3.62297×10^{-1}
1	-1.57215×10^{-2}	-1.57228×10^{-2}	-1.57280×10^{-2}	-1.57177×10^{-2}
1	3.62064×10^{-1}	3.62101×10^{-1}	3.62108×10^{-1}	3.62038×10^{-1}
2	-1.57598×10^{-2}	-1.57633×10^{-2}	-1.57664×10^{-2}	-1.57561×10^{-2}
1	3.61676×10^{-1}	3.61712×10^{-1}	3.61719×10^{-1}	3.61651×10^{-1}
3	-1.58172×10^{-2}	-1.58223×10^{-2}	-1.58239×10^{-2}	-1.58134×10^{-2}
1	3.61159×10^{-1}	3.61195×10^{-1}	3.61202×10^{-1}	3.61133×10^{-1}
4	-1.58935×10^{-2}	-1.58992×10^{-2}	-1.59002×10^{-2}	-1.58896×10^{-2}
1	3.60514×10^{-1}	3.60549×10^{-1}	3.60556×10^{-1}	3.60488×10^{-1}
5	-1.59882×10^{-2}	-1.59939×10^{-2}	-1.59951×10^{-2}	-1.59843×10^{-2}
2	-2.53439×10^{-3}	-2.53507×10^{-3}	-2.53680×10^{-3}	-2.53435×10^{-3}
2	-2.04327×10^{-3}	-2.04391×10^{-3}	-2.04474×10^{-3}	-2.04323×10^{-3}
2	-2.53117×10^{-3}	-2.53200×10^{-3}	-2.53357×10^{-3}	-2.53113×10^{-3}
3	-2.04061×10^{-3}	-2.04106×10^{-3}	-2.04208×10^{-3}	-2.04058×10^{-3}
2	-2.52688×10^{-3}	-2.52787×10^{-3}	-2.52927×10^{-3}	-2.52684×10^{-3}
4	-2.03707×10^{-3}	-2.03755×10^{-3}	-2.03853×10^{-3}	-2.03704×10^{-3}
2	-2.52153×10^{-3}	-2.52252×10^{-3}	-2.52391×10^{-3}	-2.52149×10^{-3}
5	-2.03266×10^{-3}	-2.03317×10^{-3}	-2.03411×10^{-3}	-2.03263×10^{-3}

We note that we have used the uniform convergence property of the series to interchange the order of summation and integration.

The $S_1(\alpha)$ sum is a Gaussian curve with the dispersion parameter $\sigma = \sqrt{2}\, s$. The inequality $S_1(\alpha) \gg S_2(\alpha)$, $\forall \alpha \in [0, 3\sigma]$ ensures that the influence of the first integral in the beam-shape coefficients expressions is significant, so that the second integral in Eq. (20) can be neglected.

We have compared the values of some $g_{n,\mathrm{TM}}{}^{m}$'s computed in the localized approximation method by using Eq. (20), in which we have neglected the second integral. For this calculation and for the following ones, we have considered the case $w_0 = 5$ μm, $\lambda = 0.5$ μm, $x_0 = y_0 = 2$ μm, and $z_0 = 10$ μm. A quantitative comparison of the localized approximation method and addition theorem method concerning the values of the $g_{n,\mathrm{TM}}{}^{m}$ coefficients is given in Table 1. The relative errors are given in Fig. 2 for three azimuthal modes $m = 0, 1, 2$ and n ranging from 1 to 15. The cumulative relative differences for the real and imaginary parts are less than 0.045%.

$S_1(\alpha)$ can be approximated without any significant loss of accuracy by

$$S_1(\alpha) \approx \hat{S}_1(\alpha) = \frac{1}{s^2} f_{\mathrm{cor}}(\sin\alpha)\cos\alpha \exp\left(-\frac{1}{4s^2}\sin^2\alpha\right), \tag{23}$$

where $f_{\mathrm{cor}}(\sin\alpha)$ is a correction function. For a weakly focused Gaussian beam with $s < 0.01$, one can accept that

$$f_{\mathrm{cor}}(\sin\alpha) = 1. \tag{24}$$

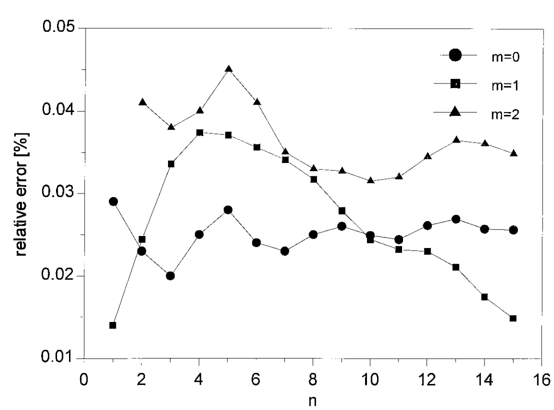

Fig. 2. Cumulative relative errors for the real and imaginary part of $g_{n,\mathrm{TM}}{}^{m}$ coefficients computed in the localized approximation method and by using Eq. (20).

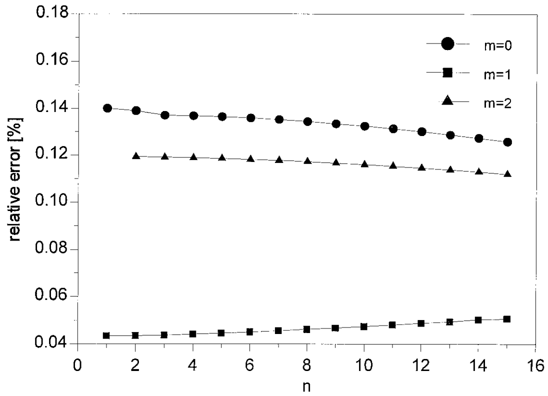

Fig. 3. Cumulative relative errors for the real and imaginary part of $g_{n,\mathrm{TM}}{}^m$ coefficients computed in the localized approximation method and by using the integral representation given in Eq. (26).

However, for a tightly focused Gaussian beam with $s < 0.1$, the correction function is given by

$$f_{\mathrm{cor}}(\sin\alpha) = 1 - s^2\left(1 - \frac{\sin\alpha}{3s}\right)\left(1 - \frac{\sin\alpha}{3\sqrt{2}\, s}\right). \quad (25)$$

With the approximation described above the beam-shape coefficients can be expressed as simple integrals over the angular coordinate α:

$$\begin{pmatrix} g_{n',\mathrm{TM}}{}^{m'} \\ ig_{n',\mathrm{TE}}{}^{m'} \end{pmatrix} = \frac{1}{4}\frac{(n'-|m'|)!}{(n'+|m'|)!} \times \int_0^{\pi/2} \exp(ik\cos\alpha z_0) \times [\Psi_{n'm'}(\alpha)T_{m'-1}(\sin\alpha) \pm \Psi_{n',-m'}(\alpha)T_{m'+1}(\sin\alpha)]\frac{1}{s^2}f_{\mathrm{cor}} \times \exp\left(-\frac{1}{4s^2}\sin^2\alpha\right)\sin\alpha\cos\alpha\mathrm{d}\alpha. \quad (26)$$

An inspection of Eq. (26) reveals that the numerical evaluation of the beam-shape coefficients requires a procedure to compute the Legendre functions and the Bessel functions of integer order.

In Fig. 3 we have plotted the relative errors between the $g_{n,\mathrm{TM}}{}^{m'}$s computed in the localized approximation method and by using the integral representation (26). In this case the relative differences are ~0.14% for $m = 0$ but ~0.05% for $m = 1$.

The exponential factor $\exp[-\sin^2\alpha/(4s^2)]$ in the integrand ensures that the interval near $\alpha = 0$ provides the largest contribution to the integral. When the approximation of the Legendre functions[21]

$$P_{n'}^{|m'|}(\cos\alpha) \to \frac{(n'+|m'|)!}{(n'-|m'|)!}\frac{J_{|m'|}[(n'+0.5)\alpha]}{(n'+0.5)^{|m'|}} \quad (27)$$

is used for $\alpha \to 0$, the estimation

$$\Psi_{n'm'}(\alpha)\sin\alpha \approx \frac{(n'-|m'|)!}{(n'+|m'|)!}\hat{\Psi}_{n'm'}(\alpha), \quad (28)$$

where

$$\hat{\Psi}_{n'm'}(\alpha) = \frac{1}{(n'+0.5)^{|m'|}}\Bigg\{(n'+1+|m'|) \times \left(\frac{n'+0.5}{n'+1.5}\right)^{|m'|} J_{|m'|}[(n'+1.5)\alpha] - \left[(n'+1)\left(1-\frac{\alpha^2}{2}\right) - m'\right]J_{|m'|}[(n+0.5)\cdot\alpha]\Bigg\} \quad (29)$$

is valid in this limit.

We thus obtain a simplified integral representation for the beam-shape coefficients:

$$\begin{pmatrix} g_{n',\mathrm{TM}}{}^{m'} \\ ig_{n',\mathrm{TE}}{}^{m'} \end{pmatrix} = \frac{\exp(ikz_0)}{4}\{U_{n'm'}\exp[i(m'-1)\varphi_0] \pm V_{n'm'}\exp[i(m'+1)\varphi_0]\}, \quad (30)$$

where $U_{n'm'}$ and $V_{n'm'}$ are integral terms expressed as

$$U_{n'm'} = \int_0^1 \hat{\Psi}_{n'm'}(x)J_{m'-1}(k\rho_0 x)\frac{1}{s^2} \times f_{\mathrm{cor}}(x)\exp\left(-\frac{x^2}{4i\bar{Q}s^2}\right)\mathrm{d}x,$$

$$V_{n'm'} = \int_0^1 \hat{\Psi}_{n',-m'}(x)J_{m'+1}(k\rho_0 x)\frac{1}{s^2} \times f_{\mathrm{cor}}(x)\exp\left(-\frac{x^2}{4i\bar{Q}s^2}\right)\mathrm{d}x. \quad (31)$$

Equation (30) is our final result for the beam-shape coefficients. We note that the integrands in Eq. (13) contain only Bessel functions of integer orders and real arguments. The beam-shape coefficients satisfy the symmetry relations

$$\begin{aligned} g_{n',\mathrm{TM}}{}^{-m'}(x_0, y_0, z_0) &= g_{n',\mathrm{TM}}{}^{m'}(x_0, -y_0, z_0), \\ g_{n',\mathrm{TE}}{}^{m'}(x_0, y_0, z_0) &= (-i)^{m'}g_{n',\mathrm{TM}}{}^{m'}(y_0, -x_0, z_0), \\ &\text{for } m' \geq 0 \\ g_{n',\mathrm{TE}}{}^{-m'}(x_0, y_0, z_0) &= -g_{n',\mathrm{TE}}{}^{m'}(x_0, -y_0, z_0), \quad \forall\, m'. \end{aligned} \quad (32)$$

Integral terms $U_{n'm'}$ and $V_{n'm'}$ can be computed from one another by using the symmetry properties

$$V_{n'm'} = (-1)^{m'+1}U_{n',-m'}, \quad (33)$$

$$U_{n'm'} = (-1)^{m'-1}V_{n',-m'}, \quad m' \geq 0. \quad (33)$$

The relative differences between the values of the $g_{n,\mathrm{TM}}{}^m$ coefficients computed in the localized approximation method and by using the simplified integral representation (30) are less than 0.06%. In view of Fig. 4, the agreement between the representations of the two beam-shape coefficients may be

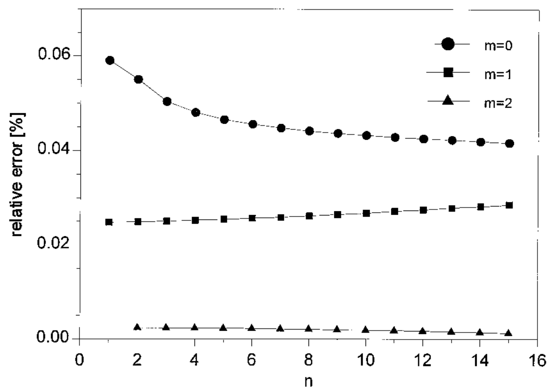

Fig. 4. Cumulative relative errors for the real and imaginary part of $g_{n,\mathrm{TM}}{}^m$ coefficients computed in the localized approximation method and by using the simplified integral representation given in Eq. (30).

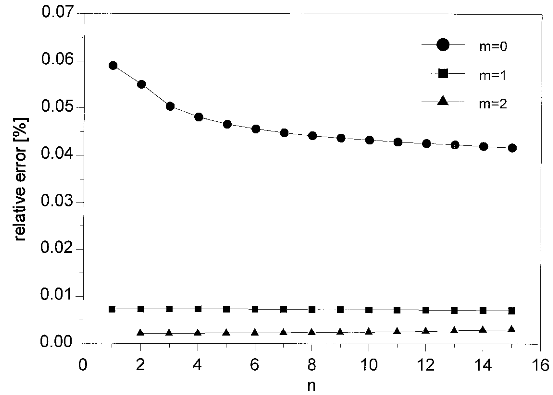

Fig. 5. Cumulative relative errors for the real and imaginary part of integral $U_{n'm'}$ and the right-hand side of Eq. (37).

considered to be perfect. It was surprising to find that the simplified integral representation is more closely related to the localized approximation than the integral representation (26). This numerical result suggests that it will be possible to provide a justification of the localized approximation starting from Eq. (30).

The most important advantage of the simplified integral representation is that by using the series expansion of the Bessel functions and the basic integral[22]

$$\int_0^\infty \exp(-a^2t^2)t^{\mu-1}J_\nu(bt) = \frac{\Gamma\left(\dfrac{\nu+\mu}{2}\right)}{\Gamma(\nu+1)}\frac{1}{2a^\mu}\left(\frac{1}{2}\frac{b}{a}\right)^\nu M \times\left(\frac{\nu+\mu}{2}, \nu+1, -\frac{b^2}{4a^2}\right), \tag{34}$$

one can analytically perform the integration to get a series representation of the beam-shape coefficients in terms of the confluent hypergeometric functions $M(\alpha, \beta, x)$.

For an on-axis Gaussian beam one can use

$$(n'+2)M[1, 2, -s^2(n'+1.5)^2] - n'M \times[1, 2, -s^2(n'+0.5)^2] \approx 2\exp[-s^2(n'+0.5)^2] \tag{35}$$

to obtain

$$\begin{aligned} g_{n',\mathrm{TM}}{}^1 &\approx g_{n',\mathrm{TM}}{}^{1,\mathrm{loc}} = 0.5g_{n'}, \\ g_{n',\mathrm{TM}}{}^{-1} &\approx g_{n',\mathrm{TM}}{}^{-1,\mathrm{loc}} = 0.5g_{n'}, \\ g_{n',\mathrm{TE}}{}^1 &\approx g_{n',\mathrm{TE}}{}^{1,\mathrm{loc}} = -0.5ig_{n'}, \\ g_{n',\mathrm{TE}}{}^{-1} &\approx g_{n',\mathrm{TE}}{}^{-1,\mathrm{loc}} = 0.5ig_{n'}, \\ g_{n'} &= i\bar{Q}\exp[-i\bar{Q}s^2(n'+0.5)^2]\exp(ikz_0), \end{aligned} \tag{36}$$

which verifies the localized approximation on axis for $z_0 \neq 0$.

This procedure can be followed for off-axis Gaussian beams, but we give a more direct justification. The form of Eqs. (17) and (30) indicates that the equality of the beam-shape coefficients computed in the localized approximation and by using the addition theorem is ensured if and only if

$$U_{n'm'} = 2(-1)^{m'-1}K_{n'm'}\bar{\Psi}_0{}^0J_{m'-1}\left(2\frac{\bar{Q}\rho_0\rho_{n'}}{w_0{}^2}\right). \tag{37}$$

If Eq. (37) is valid, we obtain

$$V_{n'm'} = 2(-1)^{m'-1}K_{n'm'}\bar{\Psi}_0{}^0J_{m'+1}\left(2\frac{\bar{Q}\rho_0\rho_{n'}}{w_0{}^2}\right), \tag{38}$$

according to the symmetry relations (33).

A numerical evaluation of the integral $U_{n'm'}$ and of the right-hand side of Eq. (37) shows that the relative differences are ~0.06% for $m = 0$ and smaller than 0.01% for the other azimuthal modes. From the numerical results that are plotted in Fig. 5 we may conclude that the agreement is perfect.

5. Conclusions

A description of a Gaussian beam in the particle-location system can be given by using the expression of the electromagnetic component of the incident field in the beam coordinate system and an addition theorem for the SVWF under a translation of the coordinate origin. Starting from an integral representation of the translation coefficients, we derived several integral representations for the $g_n{}^m$ coefficients. The values obtained from the integral representation computations agree with the values obtained in the localized approximation for off-axis Gaussian beams. By using a simplified integral representation for the beam-shape coefficients we have proved a justification of the localized approximation for on-axis and off-axis Gaussian beams. The most important result is, in our opinion, the identity, except for an insignificant factor, between the integral representation obtained in the addition theorem method and in a generalized plane-wave spectrum method.[23]

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