2.4 INVERSE THEORY: PROBLEM ON THE LINE[1]

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Keywords: 1-D Schrödinger equation, Inverse scattering on the line, Faddeev-Marchenko method, Gel'fand-Levitan method, Newton-Marchenko method, D-bar method, Trace method, Jost solutions, Scattering matrix, Wiener-Hopf factorization, Riemann-Hilbert problem, Characterization of the scattering data, Phase recovery, Steplike potentials, Neutron reflectometry, Nonhomogeneous media with jump discontinuities

2.4.1. SCHRÖDINGER EQUATION

Consider the one-dimensional Schrödinger equation

$$\frac{d^2\psi(k,x)}{dx^2} + k^2\,\psi(k,x) = V(x)\,\psi(k,x), \qquad x \in \mathbf{R}, \tag{1.1}$$

with a real-valued potential V belonging to $L^1_1(\mathbf{R})$, where $L^1_\beta(I)$ denotes the class of measurable potentials such that $\int_I dx\,(1+|x|)^\beta\,|V(x)|$ is finite. The analysis of (1.1) is fundamental for the understanding of the direct and inverse scattering problems for many other related equations in one dimension. The direct scattering problem is the problem of finding appropriate solutions to (1.1) which can be used to describe the scattering process associated with the time-dependent Schrödinger equation. The solution of (1.1) allows one to identify a certain set of data, the *scattering data,* which describes characteristic features

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[1] This work was supported in part by the National Science Foundation under grant DMS-9803219.

of the scattering process such as reflection and transmission. The inverse scattering problem, on the other hand, deals with the construction of V using the scattering data. We refer to those solutions of (1.1) that behave like e^{ikx} or e^{-ikx} as $x \to \pm\infty$ for $k \in \mathbf{R} \setminus \{0\}$ as the *scattering solutions.* The *bound states* of (1.1) correspond to the square-integrable solutions, and they occur only at certain k values on the positive imaginary axis in the complex plane $\mathbf{C}$. The analysis of (1.1) at $k = 0$ requires a separate discussion.

There are several aspects of the inverse problem. The *reconstruction problem* is the problem of finding procedures to recover the potential from the scattering data. The *characterization problem* is about the identification of necessary and sufficient conditions on the scattering data ensuring that the reconstructed potential belongs to a particular class. The *stability problem* consists of analyzing how the constructed potential changes when the scattering data is perturbed. We will not discuss the stability problem here and refer the reader to (Cha 89) and the references therein. Since inverse problems are in general ill posed, the aim is to find appropriate restrictions on the scattering data or on the potential so that the inverse scattering problem is well posed.

Scattering solutions and scattering coefficients

Among the scattering solutions of (1.1) are the Jost solution from the left, f_l, and the Jost solution from the right, f_r, satisfying the boundary conditions

$$e^{-ikx} f_l(k,x) = 1 + o(1), \quad e^{-ikx} f_l'(k,x) = ik + o(1), \qquad x \to +\infty, \tag{1.2}$$

$$e^{ikx} f_r(k,x) = 1 + o(1), \quad e^{ikx} f_r'(k,x) = -ik + o(1), \qquad x \to -\infty, \tag{1.3}$$

where the prime denotes the derivative with respect to the spatial coordinate x. From the asymptotics

$$f_l(k,x) = \frac{e^{ikx}}{T(k)} + \frac{L(k)\, e^{-ikx}}{T(k)} + o(1), \qquad x \to -\infty, \tag{1.4}$$

$$f_r(k,x) = \frac{e^{-ikx}}{T(k)} + \frac{R(k)\, e^{ikx}}{T(k)} + o(1), \qquad x \to +\infty, \tag{1.5}$$

we obtain the *scattering coefficients,* namely, the *transmission coefficient* T, and the *reflection coefficients* L and R from the left and right, respectively. Since V decays to zero as $x \to \pm\infty$ in the sense of being in $L^1_1(\mathbf{R})$, the same transmission coefficient appears in (1.4) and (1.5).

We use $\mathbf{C}^+$ to denote the upper half complex plane and $\overline{\mathbf{C}^+} := \mathbf{C}^+ \cup \mathbf{R}$; similarly, $\mathbf{C}^-$ denotes the lower half complex plane and $\overline{\mathbf{C}^-} := \mathbf{C}^- \cup \mathbf{R}$. Let $[f; g] := fg' - f'g$ denote the Wronskian of f and g, and recall that the Wronskian of any two solutions of (1.1) is independent of x. For each fixed $x \in \mathbf{R}$, $f_l(\cdot, x)$ and $f_r(\cdot, x)$ have extensions from $k \in \mathbf{R}$ to $k \in \mathbf{C}^+$ such that f_l, f_r, f'_l, and f'_r are analytic in $k \in \mathbf{C}^+$ and continuous in $k \in \overline{\mathbf{C}^+}$. The scattering coefficients can also be obtained from the Wronskian relations

$$\frac{2ik}{T(k)} = [f_r(k,x); f_l(k,x)], \qquad k \in \overline{\mathbf{C}^+}, \tag{1.6}$$

$$\frac{2ik\, L(k)}{T(k)} = [f_l(k,x); f_r(-k,x)], \qquad k \in \mathbf{R}, \tag{1.7}$$

$$\frac{2ik\, R(k)}{T(k)} = [f_l(-k,x); f_r(k,x)], \qquad k \in \mathbf{R}. \tag{1.8}$$

The left hand side in (1.6) is analytic in $k \in \mathbf{C}^+$. In general, $f_l(\cdot, x)$ and $f_r(\cdot, x)$ do not have analytic extensions from $\mathbf{R}$ to $\mathbf{C}^-$, and hence the reflection coefficients L and R cannot be extended as analytic functions from $\mathbf{R}$ into $\mathbf{C}$. However, if V vanishes on the left half line $\mathbf{R}^-$, then $f_r(\cdot, x)$ has an analytic extension to $\mathbf{C}^-$ and hence the left hand side of (1.7) can be extended analytically to $\mathbf{C}^+$. Similarly, if V vanishes on the right half line $\mathbf{R}^+$, then $f_l(\cdot, x)$ has an analytic extension to $\mathbf{C}^-$ and hence the left hand side of (1.8) can be extended analytically to $\mathbf{C}^+$.

In view of (1.2), (1.3), and the fact that V is real valued, we have

$$f_l(-k^*, x) = f_l(k,x)^*, \quad f_r(-k^*, x) = f_r(k,x)^*, \qquad k \in \overline{\mathbf{C}^+}, \tag{1.9}$$

where the asterisk denotes complex conjugation. Consequently

$$T(-k) = T(k)^*, \quad R(-k) = R(k)^*, \quad L(-k) = L(k)^*, \qquad k \in \mathbf{R}. \tag{1.10}$$

We also have

$$R(k)\, T(k)^* = -L(k)^*\, T(k), \qquad k \in \mathbf{R}, \tag{1.11}$$

$$|T(k)|^2+|L(k)|^2=1=|T(k)|^2+|R(k)|^2, \qquad k\in\mathbf{R}. \tag{1.12}$$

Thus, the scattering coefficients cannot exceed one in absolute value for real k. Furthermore, from (1.6) it follows that $T(k)\neq 0$ if $k\in\mathbf{R}\backslash\{0\}$, and hence the reflection coefficients are strictly less than one in absolute value when $k\in\mathbf{R}\setminus\{0\}$. For large k one has

$$T(k)=1+O(1/k), \qquad k\to\infty \quad \text{in} \quad \overline{\mathbf{C}^+},$$

$$R(k)=o(1/k), \quad L(k)=o(1/k), \qquad k\to\pm\infty.$$

Bound states

When $V\in L^1_1(\mathbf{R})$, T is meromorphic in $\mathbf{C}^+$ and the number of its poles is finite; each such pole is simple, occurs on the positive imaginary axis, and corresponds to a bound state of V. Conversely, each bound state of V corresponds to a pole of T in $\mathbf{C}^+$. Let N indicate the number of bound states, and use $k=i\kappa_j$ with $0<\kappa_1<\cdots<\kappa_N$ to denote the bound states. At the bound states the two Jost solutions are linearly dependent and

$$\frac{f_l(i\kappa_j,x)}{f_r(i\kappa_j,x)}=\gamma_j, \qquad x\in\mathbf{R}, \tag{1.13}$$

for some real nonzero *dependency constants* γ_j. From (1.2) and (1.3) it is seen that

$$f_l(i\kappa_j,x)=e^{-\kappa_j x}\,[1+o(1)], \qquad x\to+\infty, \tag{1.14}$$

$$f_r(i\kappa_j,x)=e^{\kappa_j x}\,[1+o(1)], \qquad x\to-\infty, \tag{1.15}$$

and hence the bound-state solutions of (1.1) decay exponentially as $x\to\pm\infty$. The residue of T at $k=i\kappa_j$ is found to be

$$\text{Res}\,(T,i\kappa_j)=i\left[\int_{-\infty}^{\infty}dx\,f_l(i\kappa_j,x)\,f_r(i\kappa_j,x)\right]^{-1}. \tag{1.16}$$

The *bound-state norming constants* $c_{l;j}$ and $c_{r;j}$ are defined as

$$c_{l;j}:=\left[\int_{-\infty}^{\infty}dx\,f_l(i\kappa_j,x)^2\right]^{-1/2}, \quad c_{r;j}:=\left[\int_{-\infty}^{\infty}dx\,f_r(i\kappa_j,x)^2\right]^{-1/2}, \tag{1.17}$$

and they are related to the dependency constants via the residues of T by

$$\operatorname{Res}(T, i\kappa_j) = i\, c_{l;j}^2\, \gamma_j = i\, \frac{c_{r;j}^2}{\gamma_j}. \tag{1.18}$$

When $\beta \in (0, +\infty)$, the quantity $1/T(i\beta)$ is real, continuous, has simple zeros at $\beta = \kappa_j$, and behaves like $1 + O(1/\beta)$ as $\beta \to +\infty$. Thus, $(-1)^N T(i\beta) > 0$ when $\beta \in (0, \kappa_1)$, $T(i\beta) > 0$ when $\beta > \kappa_N$, and $(-1)^{N-j+1} T(i\beta) > 0$ when $\beta \in (\kappa_{j-1}, \kappa_j)$ for $j = 2, \dots, N$. Consequently, $\gamma_j = (-1)^{N-j} c_{r;j}/c_{l;j}$ and

$$\operatorname{Res}(T, i\kappa_j) = (-1)^{N-j} i\, c_{l;j}\, c_{r;j}. \tag{1.19}$$

Zero-energy behavior in the L_1^1 class

In considering the small-k asymptotics of the scattering coefficients, it is necessary to distinguish between the *generic* case and the *exceptional* case. Let us define

$$W_0 := [f_r(0,x); f_l(0,x)]. \tag{1.20}$$

In the generic case, $f_l(0,\cdot)$ and $f_r(0,\cdot)$ are linearly independent on $\mathbf{R}$, and W_0 is a real nonzero constant. In the exceptional case, $f_l(0,\cdot)$ and $f_r(0,\cdot)$ are linearly dependent, $W_0 = 0$, and

$$\frac{f_l(0,x)}{f_r(0,x)} = \gamma_0, \qquad x \in \mathbf{R}, \tag{1.21}$$

for some real nonzero constant γ_0. Thus, for exceptional potentials, the zero-energy Jost solutions are bounded on $\mathbf{R}$.

When $V \in L_1^1(\mathbf{R})$, the scattering coefficients T, L, and R are all continuous at $k = 0$ (Kla 88). As $k \to 0$ in $\overline{\mathbf{C}^+}$ one has

$$T(k) = \begin{cases} \dfrac{2ik}{W_0} + o(k), & \text{generic case,} \\[2ex] \dfrac{2\gamma_0}{\gamma_0^2 + 1} + o(1), & \text{exceptional case.} \end{cases} \tag{1.22}$$

As $k \to 0$ in $\mathbf{R}$ one has

$$L(k) = \begin{cases} -1 + o(1), & \text{generic case,} \\[2ex] \dfrac{\gamma_0^2 - 1}{\gamma_0^2 + 1} + o(1), & \text{exceptional case,} \end{cases} \tag{1.23}$$

$$R(k)=\begin{cases} -1+o(1), & \text{generic case,} \\ \\ \dfrac{1-\gamma_0^2}{\gamma_0^2+1}+o(1), & \text{exceptional case.} \end{cases} \tag{1.24}$$

If $V \in L^1_2(\mathbf{R})$, then the scattering coefficients are differentiable at $k=0$. If $|V(x)| \leq Ce^{-\alpha|x|}$ for some $C \geq 0$ and $\alpha > 0$, then the scattering coefficients are analytic at $k=0$ and the coefficients in the corresponding Maclaurin series are known (Bol 85, Bol 87).

When $V \in L^1_1(\mathbf{R})$, the number of bound states is related to the argument of T via the Levinson theorem

$$\arg T(0^+) = \left(N - \frac{d}{2}\right)\pi, \tag{1.25}$$

where arg denotes the continuous branch of the argument function normalized such that $\arg T(+\infty)=0$. Here $d=1$ if V is generic and $d=0$ if V is exceptional. If $V \in L^1(\mathbf{R})$ but $V \notin L^1_1(\mathbf{R})$, then the poles of T may accumulate at $k=0$ and hence the number of bound states may be infinite; this makes the recovery of V difficult, and in fact there are no general inversion methods for potentials in $L^1(\mathbf{R})$.

Scattering matrix

The scattering matrix is given by

$$\mathbf{S}(k) := \begin{bmatrix} T(k) & R(k) \\ L(k) & T(k) \end{bmatrix}, \qquad k \in \mathbf{R}, \tag{1.26}$$

and it can be constructed in terms of the bound-state energies and either one of the reflection coefficients R and L. Given $R(k)$ for $k \in \mathbf{R}$ and the bound-state poles $k = i\kappa_j$, one can construct T and L as

$$T(k) = \left(\prod_{j=1}^{N} \frac{k+i\kappa_j}{k-i\kappa_j}\right) \exp\left(\frac{1}{2\pi i}\int_{-\infty}^{\infty} dt\, \frac{\log(1-|R(t)|^2)}{t-k-i0^+}\right), \qquad k \in \overline{\mathbf{C}^+}, \tag{1.27}$$

$$L(k) = -\frac{R(k)^*\, T(k)}{T(k)^*}, \qquad k \in \mathbf{R}. \tag{1.28}$$

Similarly, given $L(k)$ for $k \in \mathbf{R}$ and the bound-state poles $k = i\kappa_j$, one can construct T and R as

$$T(k) = \left(\prod_{j=1}^{N} \frac{k+i\kappa_j}{k-i\kappa_j}\right) \exp\left(\frac{1}{2\pi i}\int_{-\infty}^{\infty} dt\, \frac{\log(1-|L(t)|^2)}{t-k-i0^+}\right), \qquad k \in \overline{\mathbf{C}^+}, \tag{1.29}$$

$$R(k) = -\frac{L(k)^* \, T(k)}{T(k)^*}, \qquad k \in \mathbf{R}. \tag{1.30}$$

Darboux transformation

The Darboux transformation allows us to add to a given potential or to remove from it any number of bound states. It provides an alternative method to deal with bound states in the reconstruction of a potential.

Let us use a tilde to denote the quantities associated with the resulting Schrödinger equation when a bound state is added to (1.1) at $k = i\kappa$ with $\kappa > \kappa_N$ (with $\kappa > 0$ if (1.1) has no bound states). That is, $\tilde{V}$ is the resulting potential, $\tilde{T}$, $\tilde{L}$, and $\tilde{R}$ are the scattering coefficients, and $\tilde{f}_l$ and $\tilde{f}_r$ are the Jost solutions from the left and from the right, respectively. We have

$$\tilde{V}(x; \kappa, \alpha) = V(x) - 2\, \chi'(x; \kappa, \alpha), \tag{1.31}$$

$$\tilde{f}_l(k, x; \kappa, \alpha) = \frac{1}{i(k + i\kappa)} \left[f_l'(k, x) - \chi(x; \kappa, \alpha)\, f_l(k, x) \right], \tag{1.32}$$

$$\tilde{f}_r(k, x; \kappa, \alpha) = \frac{i}{(k + i\kappa)} \left[f_r'(k, x) - \chi(x; \kappa, \alpha)\, f_r(k, x) \right], \tag{1.33}$$

$$\tilde{T}(k; \kappa, \alpha) = \frac{k + i\kappa}{k - i\kappa}\, T(k), \tag{1.34}$$

$$\tilde{L}(k; \kappa, \alpha) = -\frac{k + i\kappa}{k - i\kappa}\, L(k), \quad \tilde{R}(k; \kappa, \alpha) = -\frac{k + i\kappa}{k - i\kappa}\, R(k), \tag{1.35}$$

where

$$\chi(x; \kappa, \alpha) := \frac{f_l'(i\kappa, x) + \alpha\, f_r'(i\kappa, x)}{f_l(i\kappa, x) + \alpha\, f_r(i\kappa, x)}, \qquad x \in \mathbf{R}, \tag{1.36}$$

and α corresponds to the dependency constant at the bound state $k = i\kappa$, i.e.

$$\alpha = \frac{\tilde{f}_l(i\kappa, x; \kappa, \alpha)}{\tilde{f}_r(i\kappa, x; \kappa, \alpha)}. \tag{1.37}$$

The function $\chi'(\cdot; \kappa, \alpha)$ belongs to $L^1_1(\mathbf{R})$, and hence $\tilde{V} \in L^1_1(\mathbf{R})$ whenever $V \in L^1_1(\mathbf{R})$.

Conversely, assume that $\tilde{V}$ is real valued and belongs to $L^1_1(\mathbf{R})$ and that its lowest bound-state energy corresponds to $k = i\kappa$ for some $\kappa > 0$. Let $\tilde{f}_l$ and $\tilde{f}_r$ denote the Jost solutions for $\tilde{V}$, from the left and from the right, respectively. After the removal of the bound state at $k = i\kappa$, let us denote the resulting potential by V with the corresponding Jost solutions f_l and f_r. Then we get

$$V(x) = \tilde{V}(x) - 2\,\eta'(x),$$

$$f_l(k,x) = \frac{1}{i(k-i\kappa)}\left[\tilde{f}_l'(k,x) - \eta(x)\,\tilde{f}_l(k,x)\right],$$

$$f_r(k,x) = \frac{i}{(k-i\kappa)}\left[\tilde{f}_r'(k,x) - \eta(x)\,\tilde{f}_r(k,x)\right],$$

where $\eta(x) := \tilde{f}_l'(i\kappa,x)/\tilde{f}_l(i\kappa,x)$.

In recovering V from the scattering data, it is possible to remove all the bound states from the data first and construct the resulting potential $V^{[0]}$ corresponding to the scattering coefficients $T^{[0]}$, $R^{[0]}$, $L^{[0]}$. Then the bound-state information can be used as in (1.31)-(1.37) to construct V. We have

$$T(k) = T^{[0]}(k)\prod_{j=1}^{N}\frac{k+i\kappa_j}{k-i\kappa_j}, \tag{1.38}$$

$$L(k) = (-1)^N L^{[0]}(k)\prod_{j=1}^{N}\frac{k+i\kappa_j}{k-i\kappa_j}, \quad R(k) = (-1)^N R^{[0]}(k)\prod_{j=1}^{N}\frac{k+i\kappa_j}{k-i\kappa_j}. \tag{1.39}$$

The potential $V^{[0]}$ belongs to $L_1^1(\mathbf{R})$ whenever $V \in L_1^1(\mathbf{R})$.

Fragmentation of the potential

By partitioning the real axis as $\mathbf{R} = \cup_{j=1}^{p}(x_{j-1}, x_j)$ with $x_0 := -\infty$, $x_p = +\infty$, and $x_{j-1} < x_j$ for $j = 1, \dots, p$, we obtain a fragmentation of the potential as $V(x) = \sum_{j=1}^{p} V_j(x)$, where

$$V_j(x) := \begin{cases} V(x), & x \in (x_{j-1}, x_j), \\ 0, & \text{elsewhere.} \end{cases}$$

Let N_j, T_j, R_j, and L_j denote the number of bound states, the transmission coefficient, and the reflection coefficients from the right and left, respectively, for the potential V_j. The number of bound states for V satisfies, see e.g. (Akt 98b),

$$1 - p + \sum_{j=1}^{p} N_j \le N \le \sum_{j=1}^{p} N_j,$$

and the scattering coefficients for V can be expressed in terms of those for V_j as

$$\begin{bmatrix} \dfrac{1}{T(k)} & -\dfrac{R(k)}{T(k)} \\ \dfrac{L(k)}{T(k)} & \dfrac{1}{T(-k)} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{T_1(k)} & -\dfrac{R_1(k)}{T_1(k)} \\ \dfrac{L_1(k)}{T_1(k)} & \dfrac{1}{T_1(-k)} \end{bmatrix} \cdots \begin{bmatrix} \dfrac{1}{T_p(k)} & -\dfrac{R_p(k)}{T_p(k)} \\ \dfrac{L_p(k)}{T_p(k)} & \dfrac{1}{T_p(-k)} \end{bmatrix}, \qquad k \in \mathbf{R}. \tag{1.40}$$

Factorizations of the form (1.40) are helpful in the recovery of potentials that are partially known.

2.4.2. METHODS TO SOLVE THE INVERSE SCATTERING PROBLEM

When there are no bound states, either one of the reflection coefficients R and L uniquely determines a real-valued potential V in $L_1^1(\mathbf{R})$. However, when there are bound states, for the unique determination of V, in addition to one reflection coefficient and the bound-state energies, one must also specify the bound-state norming constant or, equivalently, the dependency constant for each bound state. As our scattering data we can choose either the left scattering data $\{R, \{\kappa_j\}, \{c_{l;j}\}\}$ or the right scattering data $\{L, \{\kappa_j\}, \{c_{r;j}\}\}$. These two are equivalent to each other, and from (1.18) and (1.26)-(1.30) it is seen that each is equivalent to $\{\mathbf{S}, \{\gamma_j\}\}$.

Riemann-Hilbert problems

Since k appears as k^2 in (1.1), the functions $f_l(-k,x)$ and $f_r(-k,x)$ are also solutions of (1.1) and they can be expressed as linear combinations of the Jost solutions $f_l(k,x)$ and $f_r(k,x)$ for real k as

$$f_l(-k,x) = T(k)\, f_r(k,x) - R(k)\, f_l(k,x), \qquad k \in \mathbf{R},$$

$$f_r(-k,x) = T(k)\, f_l(k,x) - L(k)\, f_r(k,x), \qquad k \in \mathbf{R},$$

or equivalently as

$$m_l(-k,x) = T(k)\, m_r(k,x) - R(k)\, e^{2ikx}\, m_l(k,x), \qquad k \in \mathbf{R}, \tag{2.1}$$

$$m_r(-k,x) = T(k)\, m_l(k,x) - L(k)\, e^{-2ikx}\, m_r(k,x), \qquad k \in \mathbf{R}, \tag{2.2}$$

where m_l and m_r are the *Faddeev functions* defined as

$$m_l(k,x) := e^{-ikx} f_l(k,x), \quad m_r(k,x) := e^{ikx} f_r(k,x). \tag{2.3}$$

Each of (2.1) and (2.2) can be viewed as a *Riemann-Hilbert problem* (New 83, Abl 91), where, given the scattering coefficients for $k \in \mathbf{R}$, the aim is to construct m_l and m_r such that, for each $x \in \mathbf{R}$, $m_l(\cdot, x)$ and $m_r(\cdot, x)$ are analytic in $\mathbf{C}^+$, continuous in $\overline{\mathbf{C}^+}$, and behave like $1 + O(1/k)$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$.

Faddeev-Marchenko method

In this method (Fad 67, Dei 79, Cha 89) the potential is constructed from the left scattering data $\{R, \{\kappa_j\}, \{c_{l;j}\}\}$ by solving the left Marchenko integral equation or from the right scattering data $\{L, \{\kappa_j\}, \{c_{r;j}\}\}$ by solving the right Marchenko integral equation.

The left Marchenko equation using the left scattering data as the input is given by

$$B_l(x,\alpha) = g_l(2x+\alpha) + \int_0^\infty d\beta \, g_l(2x+\alpha+\beta)\, B_l(x,\beta), \qquad \alpha > 0, \tag{2.4}$$

where

$$g_l(\alpha) := -\hat{R}(\alpha) - \sum_{j=1}^{N} c_{l;j}^2 \, e^{-\kappa_j \alpha}. \tag{2.5}$$

with

$$\hat{R}(\alpha) := \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, R(k)\, e^{ik\alpha}. \tag{2.6}$$

One can obtain (2.4) from (2.1) by using a Fourier transformation. The potential is obtained from the solution $B_l(x,\alpha)$ as

$$V(x) = -2\, \frac{dB_l(x, 0^+)}{dx}, \tag{2.7}$$

and the Jost solution from the left is constructed as

$$f_l(k,x) = e^{ikx} \left[1 + \int_0^\infty d\alpha \, B_l(x,\alpha)\, e^{ik\alpha} \right]. \tag{2.8}$$

Similarly, via a Fourier transformation of (2.2), using the right scattering data as the input one obtains the right Marchenko equation

$$B_r(x,\alpha) = g_r(-2x+\alpha) + \int_0^\infty d\beta\, g_r(-2x+\alpha+\beta)\, B_r(x,\beta), \qquad \alpha > 0, \tag{2.9}$$

where

$$g_r(\alpha) := -\hat{L}(\alpha) - \sum_{j=1}^N c_{r;j}^2\, e^{-\kappa_j \alpha}, \tag{2.10}$$

with

$$\hat{L}(\alpha) := \frac{1}{2\pi}\int_{-\infty}^\infty dk\, L(k)\, e^{ik\alpha}. \tag{2.11}$$

The potential is recovered by using

$$V(x) = 2\,\frac{dB_r(x,0^+)}{dx}, \tag{2.12}$$

and the Jost solution from the right is constructed as

$$f_r(k,x) = e^{-ikx}\left[1 + \int_0^\infty d\alpha\, B_r(x,\alpha)\, e^{ik\alpha}\right]. \tag{2.13}$$

The Marchenko equations are often written in a different form. Letting

$$K_+(x,y) := B_l(x, y-x), \quad c_+ := c_{l;j}^2,$$

$$M_+(y) := -g_l(y) = \hat{R}(y) + \sum_{j=1}^N c_+\, e^{-\kappa_j y},$$

the left Marchenko equation (2.4) takes the form

$$K_+(x,y) + M_+(x+y) + \int_x^\infty dz\, M_+(y+z)\, K_+(x,z) = 0, \qquad y > x.$$

Once this Marchenko equation is solved for $K_+(x,y)$, the potential is obtained as

$$V(x) = -2\,\frac{dK_+(x,x^+)}{dx},$$

and the Jost solution from the left is constructed as

$$f_l(k,x) = e^{ikx} + \int_x^\infty dy\, K_+(x,y)\, e^{iky}.$$

Similarly, by letting

$$K_-(x,y) := B_r(x, x-y), \quad c_- := c_{r;j}^2,$$

$$M_-(y) := -g_r(-y) = \hat{L}(-y) + \sum_{j=1}^{N} c_- \, e^{\kappa_j y},$$

the right Marchenko equation (2.9) takes the form

$$K_-(x,y) + M_-(x+y) + \int_{-\infty}^{x} dz \, M_-(y+z) \, K_-(x,z) = 0, \qquad y < x.$$

Once this Marchenko equation is solved for $K_-(x,y)$, the potential is obtained as

$$V(x) = 2 \, \frac{dK_-(x, x^-)}{dx},$$

and the Jost solution from the right is constructed as

$$f_r(k,x) = e^{-ikx} + \int_{-\infty}^{x} dy \, K_-(x,y) \, e^{-iky}.$$

Gel'fand-Levitan method

In this method (New 83, Lev 87) the potential is constructed from the corresponding *spectral function,* which is a 2×2 real-valued matrix function of energy $E = k^2$.

The Gel'fand-Levitan equation is given by

$$h(x,\alpha) = \nu(\alpha, x) - \int_{-x}^{x} d\beta \, \nu(\alpha,\beta) \, h(x,\beta), \qquad -|x| < \alpha < |x|,$$

where

$$\nu(\alpha,\beta) := \int_{-\infty}^{\infty} dE \, [\, e^{-ik\alpha} \quad e^{ik\alpha} \,] \, \frac{d\rho}{dE} \begin{bmatrix} e^{ik\beta} \\ e^{-ik\beta} \end{bmatrix}.$$

with the (modified) spectral function given as

$$\frac{d\rho}{dE} = \begin{cases} \dfrac{1}{4\pi k} \, \left([J(k) \, J(k)^\dagger]^{-1} - I_2 \right), & E > 0, \\[2ex] \displaystyle\sum_{j=1}^{N} M_j \, \delta(E - E_j), & E < 0. \end{cases}$$

Here I_2 is the 2×2 identity matrix, δ indicates the Dirac delta distribution, M_j are certain 2×2 constant matrices related to the bound-state data, and $J(k)$ is the *Jost matrix,* which is a 2×2 matrix-valued function of k relating the Jost solutions f_l and f_r and the *regular solutions* ϕ_l and ϕ_r as

$$T(k)\,J(k)\begin{bmatrix} f_l(k,x)\\ f_r(k,x)\end{bmatrix}=\begin{bmatrix}\phi_l(k,x)\\ \phi_r(k,x)\end{bmatrix},$$

where ϕ_l and ϕ_r are the solutions of (1.1) satisfying

$$\phi_l(k,0)=1,\quad \phi_l'(k,0)=ik,\quad \phi_r(k,0)=1,\quad \phi_r'(k,0)=-ik.$$

In the Gel'fand-Levitan method, if one wants to use as an input either the left or right scattering data instead of the spectral function, then one first needs to construct $J(k)$ and M_j from such data; the reader is referred to (New 83) for such a construction.

The potential is obtained from the solution $h(x,\alpha)$ of the Gel'fand-Levitan equation as

$$V(x)=-2\,\frac{dh(x,x^{\mp})}{dx},\qquad \pm x\geq 0,$$

and the regular solutions are constructed as

$$\begin{bmatrix}\phi_l(k,x)\\ \phi_r(k,x)\end{bmatrix}=\begin{bmatrix}e^{ikx}\\ e^{-ikx}\end{bmatrix}-\int_{-x}^{x}d\alpha\,h(x,\alpha)\begin{bmatrix}e^{ik\alpha}\\ e^{-ik\alpha}\end{bmatrix}.$$

Trace method

The details of the trace method can be found in (Dei 79). In this method one needs to obtain the Faddeev function m_l by solving the first-order nonlinear system

$$\begin{cases} m_l'(k,x)=e^{-2ikx}\,n_l(k,x),\\ n_l'(k,x)=e^{2ikx}\,m_l(k,x)\,Q_l(x),\end{cases}\tag{2.14}$$

with the boundary conditions at $x=+\infty$ given by

$$m_l(k,x)=1+o(1),\quad e^{-2ikx}\,n_l(k,x)=o(1),\qquad x\to+\infty,$$

where

$$Q_l(x) := \frac{2i}{\pi} \int_{-\infty}^{\infty} dk\, k\, R(k)\, e^{2ikx}\, m_l(k,x)^2 - 4 \sum_{j=1}^{N} \kappa_j\, c_{l;j}^2\, e^{-2\kappa_j x}\, m_l(i\kappa_j, x)^2. \tag{2.15}$$

The nonlinearity in (2.14) is due to the fact that $Q_l(x)$ depends on $m_l(k,x)$; notice that the left scattering data is used as the input in (2.15). Once this system is solved, the potential is obtained as $V(x) = Q_l(x)$. The integral in (2.15) should be interpreted as the limit of Césaro means (Dei 79) in case it does not exist in the Lebesgue sense.

One can also use the right scattering data $\{L, \{\kappa_j\}, \{c_{r;j}\}\}$ by setting up a similar first-order nonlinear system involving m_r

$$\begin{cases} m_r'(k,x) = e^{2ikx}\, n_r(k,x), \\ n_r'(k,x) = e^{-2ikx}\, m_r(k,x)\, Q_r(x), \end{cases}$$

with the boundary conditions at $x = -\infty$ given by

$$m_r(k,x) = 1 + o(1), \quad e^{2ikx}\, n_r(k,x) = o(1), \qquad x \to -\infty,$$

where

$$Q_r(x) := \frac{2i}{\pi} \int_{-\infty}^{\infty} dk\, k\, L(k)\, e^{-2ikx}\, m_r(k,x)^2 - 4 \sum_{j=1}^{N} \kappa_j\, c_{r;j}^2\, e^{2\kappa_j x}\, m_r(i\kappa_j, x)^2.$$

Once the nonlinear system is solved, the potential is obtained as $V(x) = Q_r(x)$.

Singular integral equations

From (1.13), (1.18), and (2.1), with the help of a contour integration, one finds that the Faddeev function m_l satisfies the singular integral equation

$$m_l(k,x) = 1 - i \sum_{j=1}^{N} \frac{c_{l;j}^2\, e^{-2\kappa_j x}}{k + i\kappa_j}\, m_l(i\kappa_j, x) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt\, \frac{R(t)\, e^{2itx}}{t + k + i0^+}\, m_l(t,x), \tag{2.16}$$

where the left scattering data $\{R, \{\kappa_j\}, \{c_{l;j}\}\}$ is used as the input. If one can solve this singular integral equation, then the potential can be obtained with the help of the Schrödinger equation by using

$$V(x) = \frac{1}{m_l(k,x)}\, [m_l''(k,x) + 2ik\, m_l'(k,x)]. \tag{2.17}$$

Similarly, with the help of (2.2), one finds that the Faddeev function m_r satisfies the singular integral equation

$$m_r(k,x) = 1 - i \sum_{j=1}^{N} \frac{c_{r;j}^2 \, e^{2\kappa_j x}}{k + i\kappa_j} \, m_r(i\kappa_j, x) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt \, \frac{L(t) \, e^{-2itx}}{t + k + i0^+} \, m_r(t,x), \tag{2.18}$$

where the right scattering data $\{L, \{\kappa_j\}, \{c_{r;j}\}\}$ is used as the input. Once this singular integral equation is solved for $m_r(k,x)$, the potential can be obtained by using

$$V(x) = \frac{1}{m_r(k,x)} \, [m_r''(k,x) - 2ik \, m_r'(k,x)]. \tag{2.19}$$

Newton-Marchenko method

As mentioned earlier, knowledge of $\{\mathbf{S}, \{\gamma_j\}\}$ is equivalent to that of either the left or right scattering data. In the Newton-Marchenko method the potential is constructed from the scattering data consisting of the whole scattering matrix and the bound-state dependency constants. The dependency constants are used to fix the norms of the eigenvectors of the Jost matrix at the bound-state energies. The reader is referred to (New 83, Cha 89) for details. This method has a generalization to higher dimensions because it uses the scattering matrix itself as an input to the Newton-Marchenko integral equation.

When there are no bound states, the Newton-Marchenko equation is the coupled system given by

$$\eta(x,\alpha) = H(x,\alpha) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \int_0^{\infty} d\beta \, H(x, \alpha+\beta) \, \eta(x,\beta), \qquad \alpha > 0, \tag{2.20}$$

where

$$H(x,\alpha) := \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \begin{bmatrix} T(k) - 1 & L(k) \, e^{-2ikx} \\ R(k) \, e^{2ikx} & T(k) - 1 \end{bmatrix} e^{-ik\alpha}.$$

One can obtain (2.20) from (2.1) and (2.2) by using $k \mapsto -k$, a multiplication by $T(k)$, and a Fourier transformation in succession. Once the coupled system (2.20) is solved, one has the Faddeev functions by using

$$\begin{bmatrix} T(k) \, m_l(k,x) \\ T(k) \, m_r(k,x) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \int_0^{\infty} d\alpha \, \eta(x,\alpha) \, e^{ik\alpha},$$

and the potential is obtained as

$$V(x) = -2\,\frac{d\eta(x,0^+)}{dx}\begin{bmatrix}1\\0\end{bmatrix} = 2\,\frac{d\eta(x,0^+)}{dx}\begin{bmatrix}0\\1\end{bmatrix}.$$

When there are bound states, they can first be removed by using Darboux transformations; after the inverse scattering problem is solved by the Newton-Marchenko method, the modifications due to the bound states can be implemented (New 83).

Wiener-Hopf factorization

In this method one obtains the Wiener-Hopf factorization of the unitarily dilated scattering matrix given by

$$G(k,x) := \begin{bmatrix} T(k) & -R(k)\,e^{2ikx} \\ -L(k)\,e^{-2ikx} & T(k)\end{bmatrix}, \qquad k\in\mathbf{R}.$$

Such a factorization has the form

$$G(k,x) = G_-(k,x)\,D(k)\,G_+(k,x), \qquad k\in\mathbf{R},$$

where

$$D(k) = \left(\frac{k-i}{k+i}\right)^{\rho_1}\mathbf{Q}_+ + \left(\frac{k-i}{k+i}\right)^{\rho_2}\mathbf{Q}_-.$$

For each $x\in\mathbf{R}$, the matrix function $G_\pm(\cdot,x)$ and its inverse $G_\pm(\cdot,x)^{-1}$ are continuous in $\overline{\mathbf{C}^\pm}$ and analytic in $\mathbf{C}^\pm$. Moreover, $G_\pm(k,x)\to I_2$ as $k\to\infty$ in $\overline{\mathbf{C}^\pm}$. The numbers ρ_1 and ρ_2 are integers called *partial indices* and are uniquely determined by $G(k,x)$; if both ρ_1 and ρ_2 are zero, the factorization is *canonical,* otherwise *noncanonical.* The matrices $\mathbf{Q}_\pm$ are complementary rank-one projections and can be chosen as

$$\mathbf{Q}_\pm := \frac{1}{2}\begin{bmatrix}1 & \pm1\\ \pm1 & 1\end{bmatrix}.$$

Note that the system (2.1) and (2.2) can be written as

$$\begin{bmatrix} m_l(-k,x)\\ m_r(-k,x)\end{bmatrix} = G(k,x)\begin{bmatrix} m_r(k,x)\\ m_l(k,x)\end{bmatrix}, \qquad k\in\mathbf{R}.$$

Hence, once the Wiener-Hopf factorization of $G(k,x)$ is obtained, one has essentially solved the inverse scattering problem. In the exceptional case without bound states one has $D(k)=I_2$, and the Faddeev functions m_l and m_r are obtained from $G_-(k,x)$ as

$$\begin{bmatrix} m_l(k,x) \\ m_r(k,x) \end{bmatrix} = G_-(-k,x) \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad k\in\overline{\mathbf{C}^+}.$$

In the generic case without bound states one has $\rho_1=0$ and $\rho_2=1$. The Wiener-Hopf factors $G_\pm(k,x)$ depend on a free parameter $a\in[0,+\infty]$, but that free parameter does not appear in the Faddeev functions m_l and m_r, which are recovered as

$$\begin{bmatrix} m_l(k,x) \\ m_r(k,x) \end{bmatrix} = G_-(-k,x) \left[\mathbf{Q}_+ + \left(\frac{k+i}{k}\right)\mathbf{Q}_-\right] \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad k\in\overline{\mathbf{C}^+}.$$

Having found the Faddeev functions, the potential is obtained as in (2.17) or (2.19).

In the exceptional case, when there are N bound states, the partial indices are given by $\rho_1=\rho_2=-N$. The Faddeev functions corresponding to the potential $V^{[0]}$ with the scattering coefficients $T^{[0]}$, $R^{[0]}$, and $L^{[0]}$ as in (1.38) and (1.39), are obtained as

$$\begin{bmatrix} m_l^{[0]}(k,x) \\ m_r^{[0]}(k,x) \end{bmatrix} = \left(\prod_{j=1}^{N}\frac{k+i\kappa_j}{k+i}\right) G_-(-k,x) \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad k\in\overline{\mathbf{C}^+}. \tag{2.21}$$

Even though $G_-(k,x)$ depends on $2N$ parameters, the Faddeev functions in (2.21) depend on N parameters that are uniquely determined by the N normalization constants. The potential V and the Jost solutions can be constructed as in (1.31)-(1.33) to complete the solution of the inverse scattering problem.

In the generic case, when there are N bound states, the partial indices are given by

$$\rho_1=\begin{cases} -N, & N \text{ even},\\ -N+1, & N \text{ odd}, \end{cases} \qquad \rho_2=\begin{cases} -N+1, & N \text{ even},\\ -N, & N \text{ odd}. \end{cases}$$

The Faddeev functions corresponding to the potential $V^{[0]}$ are, for $k\in\overline{\mathbf{C}^+}$, obtained as

$$\begin{bmatrix} m_l^{[0]}(k,x) \\ m_r^{[0]}(k,x) \end{bmatrix} = \begin{cases} \left(\prod_{j=1}^{N}\frac{k+i\kappa_j}{k+i}\right) G_-(-k,x) \left[\mathbf{Q}_+ + \left(\frac{k+i}{k}\right)\mathbf{Q}_-\right] \begin{bmatrix} 1 \\ 1 \end{bmatrix}, & N \text{ even},\\ \left(\prod_{j=1}^{N}\frac{k+i\kappa_j}{k+i}\right) G_-(-k,x) \left[\left(\frac{k+i}{k}\right)\mathbf{Q}_+ + \mathbf{Q}_-\right] \begin{bmatrix} 1 \\ 1 \end{bmatrix}, & N \text{ odd}. \end{cases} \tag{2.22}$$

Even though $G_-(k,x)$ depends on $2N+1$ parameters, the Faddeev functions in (2.22) depend on N parameters that are uniquely determined by the N normalization constants. Again, the potential V and the Jost solutions can be constructed as in (1.31)-(1.33) to complete the solution of the inverse scattering problem.

In fact, the Wiener-Hopf factorization of $G(k,x)$ yields not only the potential V corresponding to the scattering matrix $\mathbf{S}$, but also the potential(s) corresponding to the scattering matrix obtained by reversing the signs of the reflection coefficients. In the generic case such potentials do not belong to $L^1_1(\mathbf{R})$, but they are closely related to that class, as indicated in Section 2.4.4. The additional constants appearing in the Wiener-Hopf factors $G_\pm(k,x)$, in which there are N such factors in the exceptional case and $N+1$ of them in the generic case, can be fixed by using the additional normalization constants related to the potential(s) corresponding to the scattering matrix obtained by changing the signs of the reflection coefficients. A detailed investigation of these potentials can be found in (Cha 89, Akt 93).

D-bar method

An elementary treatment of the D-bar method in inverse scattering can be found in (Abl 91, Cha 89). Let k_R and k_I denote the real and imaginary parts, respectively, of the complex variable k and $\overline{k}$ denote the complex conjugate of k. The *D-bar derivative* of a function g is defined as

$$\frac{\partial g}{\partial \overline{k}} := \frac{1}{2}\left(\frac{\partial g}{\partial k_R} + i\frac{\partial g}{\partial k_I}\right),$$

and it measures the "departure from analyticity" because $\partial g/\partial \overline{k} = 0$ in domains where g is analytic in k. For example, one has

$$\frac{\partial}{\partial \overline{k}}\left(\frac{1}{k-k_0}\right) = \pi\,\delta(k-k_0), \tag{2.23}$$

where δ denotes the Dirac delta function on the complex plane. The *Gauss-Green formula* generalizes the Cauchy integral formula, and for $g(k)$ vanishing at infinity it provides the integral representation

$$g(k)=\frac{1}{2\pi i}\oint_{\partial D}dz\,\frac{g(z)}{z-k}+\frac{1}{2\pi i}\iint_{D}\frac{dz\wedge d\overline{z}}{z-k}\,\frac{\partial g(z)}{\partial\overline{z}},\tag{2.24}$$

where the second integral is over a domain D, the first integral is along the positively-oriented boundary ∂D of D, and $dz\wedge d\overline{z}=-2i\,dz_R\,dz_I$. The idea behind the D-bar method is to relate g and $\partial g/\partial\overline{k}$ to the scattering data so that (2.24) becomes an integral equation that can be solved for g. The variable x appears in g as a parameter, and the potential V can be recovered from the solution g.

For example, let

$$g(k)=\begin{cases}T(k)\,m_r(k,x)-1, & k_I>0,\\[2mm] m_l(-k,x)-1, & k_I\le 0,\end{cases}\tag{2.25}$$

where we have suppressed the dependence of g on x. With the help of (1.38) and (2.23), we can evaluate the D-bar derivative of g as

$$\frac{\partial g(k)}{\partial\overline{k}}=\sum_{j=1}^{N}\pi\,\mathrm{Res}(T,i\kappa_j)\,m_r(i\kappa_j,x)\,\delta(k-i\kappa_j)+[T(k)\,m_r(k,x)-m_l(-k,x)]\,\delta(k_I).\tag{2.26}$$

Using (1.13), (1.18), (2.1), and (2.3), we can write (2.26) as

$$\frac{\partial g(k)}{\partial\overline{k}}=-i\pi\sum_{j=1}^{N}c_{l;j}^2\,e^{-2\kappa_j x}\,m_l(i\kappa_j,x)\,\delta(k-i\kappa_j)+\frac{i}{2}\,R(k)\,e^{2ikx}\,m_l(k,x)\,\delta(k_I).\tag{2.27}$$

Thus, from (2.25) and (2.27) we get the D-bar equation

$$\frac{\partial g(k)}{\partial\overline{k}}=\left[\frac{i}{2}\,R(k)\,e^{2ikx}\,\delta(k_I)-i\pi\sum_{j=1}^{N}c_{l;j}^2\,e^{-2\kappa_j x}\,\delta(k-i\kappa_j)\right]\,[g(-k)+1]\,,\qquad k\in\mathbf{C},$$

with the boundary condition that $g(k)\to 0$ as $k\to\infty$ in $\mathbf{C}$. Note that by choosing D as the entire complex plane (and hence the boundary ∂D as the infinite circle centered at the origin) in (2.24) and using (2.27) in (2.24), one obtains the singular integral equation given in (2.16). Once g, and hence also the Faddeev function m_l, is obtained, V can be recovered as in (2.17).

In a similar manner, letting

$$g(k) = \begin{cases} T(k)\, m_l(k,x) - 1, & k_I > 0, \\ m_r(-k,x) - 1, & k_I \leq 0, \end{cases}$$

one obtains the D-bar equation

$$\frac{\partial g(k)}{\partial \bar{k}} = \left[\frac{i}{2}\, L(k)\, e^{-2ikx} - i\pi \sum_{j=1}^{N} c_{r;j}^2\, e^{2\kappa_j x}\, \delta(k - i\kappa_j) \right] [g(-k) + 1]\,, \qquad k \in \mathbf{C}, \quad (2.28)$$

with the boundary condition that $g(k) \to 0$ as $k \to \infty$ in $\mathbf{C}$. Note that using (2.28) in (2.24) one obtains (2.18). Once (2.28) is solved for g, one can recover V as in (2.19).

Other methods

For various other methods, such as time-domain methods, various iterative methods like layer stripping, and various approximation methods like Born approximation, the reader is referred to (Cha 89) and the references therein.

2.4.3. CHARACTERIZATION

A characterization for a specific class of potentials is to present some necessary and sufficient conditions on the scattering data which guarantee that there exists a corresponding unique potential in that class. Such conditions are usually obtained by using the Faddeev-Marchenko method. The characterization conditions can be stated for the left scattering data, for the right scattering data, or for the combination of both. For a characterization in the class of real-valued potentials belonging to $L_2^1(\mathbf{R})$, the reader is referred to (Dei 79). Some characterizations in the class of real-valued potentials belonging to $L_1^1(\mathbf{R})$ were given by Melin (Mel 85) and Marchenko (Mar 86).

Given the scattering data with the reflection coefficient $R(k)$ for $k \in \mathbf{R}$, the bound-state energies $-\kappa_j^2$ with $0 < \kappa_1 < \cdots < \kappa_N$, and the bound-state norming constants $c_{l;j}$, the following conditions (Akt 00a) form a characterization in the class of real-valued potentials in $L_1^1(\mathbf{R})$:

(i) R is continuous on $\mathbf{R}$, and $R(-k) = R(k)^*$ for $k \in \mathbf{R}$.

(ii) $|R(k)| \leq 1 - Ck^2/(1+k^2)$ on $\mathbf{R}$ for some constant $C > 0$.

(iii) $R(0) \in [-1, 1)$.

(iv) $R(k) = o(1/k)$ as $k \to \pm\infty$.

(v) The function $k/T(k)$, where $T(k)$ is given by (1.27), is continuous in $\overline{\mathbf{C}^+}$.

(vi) The functions $\hat{R}$ and $\hat{L}$ defined in (2.6) and (2.11), respectively, where $L(k)$ is obtained from (1.28), are absolutely continuous. Moreover, $\hat{R}' \in L_1^1(a, +\infty)$ and $\hat{L}' \in L_1^1(-\infty, a)$ for every $a \in \mathbf{R}$.

When these conditions are satisfied, both the left and right Marchenko equations are uniquely solvable and the right hand sides of (2.7) and (2.12) are equal. Thus, V can be obtained from either (2.7) or (2.12); moreover, V belongs to $L_1^1(\mathbf{R})$.

One can state the characterization conditions differently, especially without any reference to the continuity at $k = 0$. For example, in (iv) one can replace $o(1/k)$ by $O(1/k)$ because of (vi). Also, as in (Mar 86), one can demand (i) and (ii) only for $k \in \mathbf{R} \setminus \{0\}$ and simultaneously replace (iii) with the condition that $k[1 + R(k)]/T(k) \to 0$ as $k \to 0$ on $\mathbf{R}$.

2.4.4. SPECIAL CASES

When the potential is partially known, it may be possible to recover the unknown part of the potential without knowledge of the full scattering data. For example, one can use (1.40) to obtain the scattering data associated with the unknown part of the potential. There are also other approaches to such an inverse problem, see for example (Nov 87, Akt 93, Run 94, Gre 95, Ges 97) and the references therein.

Potentials vanishing on a half line

Assume the potential V of (1.1) is real valued, $V(x) = 0$ for $x > 0$, and $V \in L_1^1(\mathbf{R}^-)$. Then, the corresponding reflection coefficient R has a meromorphic extension to $\mathbf{C}^+$ with simple poles coinciding exactly with the poles of T. Thus, the bound-state energies are uniquely determined once $R(k)$ is known on any nontrivial interval on the real axis. Moreover, the bound-state norming constants are determined by the residues of R at such poles from $c_{l;j}^2 = -i\,\mathrm{Res}(R, i\kappa_j)$. Hence, V is uniquely determined by R alone without specifying any bound-state information. However, L alone cannot uniquely determine V, but $\{L, \{\kappa_j\}\}$ does.

A similar result holds if V is real valued, $V(x) = 0$ for $x < 0$, and $V \in L_1^1(\mathbf{R}^+)$. Then V is uniquely determined by L alone without specifying any bound-state information. However, R alone cannot uniquely determine V, but $\{R, \{\kappa_j\}\}$ does.

Rational scattering coefficients

When the scattering coefficients are rational functions of k in $\mathbf{C}$, the corresponding potentials are known as *Bargmann* potentials. Such potentials decay exponentially as $x \to \pm\infty$, or they may decay exponentially at one end of the real axis and vanish on the opposite half line. These potentials can be obtained algebraically with the help of (2.16), (2.18), and a contour integration. We refer the reader to (Cha 89) for the details of this method. Some methods from the theory of realizations of rational matrix functions (Alp 98, van der Mee 00) can also be used to obtain closed-form expressions for the potential.

Potentials with Dirac delta distributions

Assume that to the real potential V belonging to $L_1^1(\mathbf{R})$, the Dirac delta distributions are added at some points so that the resulting potential W is given by

$$W(x) = V(x) + \sum_{j=1}^{n} b_j\,\delta(x - x_j),$$

where b_j are some real nonzero constants. Instead of vanishing as $o(1/k)$ as $k \to \pm\infty$, the reflection coefficients gain some $O(1/k)$ terms such that

$$2ik\,R(k) = \sum_{j=1}^{n} b_j\,e^{-2ikx_j} + o(1), \quad 2ik\,L(k) = \sum_{j=1}^{n} b_j\,e^{2ikx_j} + o(1), \qquad k \to \pm\infty.$$

Apart from some minor modifications, the inversion methods indicated in Section 2.4.2 remain applicable to recover W.

Potentials obtained when the signs of reflection coefficients are changed

If we reverse the signs of the reflection coefficients in (1.26) corresponding to $V \in L_1^1(\mathbf{R})$ without changing the sign of the transmission coefficient, it is possible to characterize (Akt 93) the corresponding potentials. Such potentials have the form

$$U(x) = \frac{2\epsilon_+}{x^2+1}\,\theta(x) + \frac{2\epsilon_-}{x^2+1}\,\theta(-x) + Q(x),$$

where $\theta(x)$ is the Heaviside function, Q is some potential belonging to $L_1^1(\mathbf{R})$, and $\epsilon_\pm \in \{0,1\}$. The exceptional case occurs if and only if $\epsilon_+ = \epsilon_- = 0$; generically $U \notin L_1^1(\mathbf{R})$. In the generic case, one or both of the corresponding Jost solutions blow up like $O(1/k)$ as $k \to 0$, and one additional normalization constant at zero energy, in addition to the N constants for the N bound states, must be specified to determine such potentials uniquely.

2.4.5. STEPLIKE POTENTIALS

Consider (1.1) when the potential has different constant asymptotics as $x \to \pm\infty$. Without any loss of generality, it is enough to consider the case $V(x) \to 0$ as $x \to -\infty$ and $V(x) \to c^2$ as $x \to +\infty$ for some $c > 0$. To be precise, assume that V is real valued and satisfies $V \in L_1^1(\mathbf{R}^-)$ and $V - c^2 \in L_1^1(\mathbf{R}^+)$. The reader is referred to (Bus 62, Coh 85) for details. Since V has different asymptotics as $x \to \pm\infty$, the transmission coefficients from the left and from the right, T_l and T_r, are unequal but related by

$$T_r(k) = \frac{\xi}{k}\,T_l(k), \tag{5.1}$$

where $\xi := \sqrt{k^2 - c^2}$, using the branch of the (complex) square-root function with $\operatorname{Im}\xi \geq 0$. The mapping $k \mapsto \xi$ is analytic from $\mathbf{C}^+$ to itself and is continuous on $\overline{\mathbf{C}^+}$. If we use T to denote T_l, then the scattering coefficients from the left T and L are defined as in (1.4) or in (1.6) and (1.7). One needs to modify (1.5), (1.8), (1.12), (1.14), and (1.22)-(1.24); the remaining displayed formulas in (1.2)-(1.25) remain unchanged. The potential V can be obtained by using the right Marchenko equation as in (2.9)-(2.13) together with the right scattering data $\{L, \{\kappa_j\}, \{c_{r;j}\}\}$. The left Marchenko equation is still given by (2.4), but the formulas (2.5)-(2.8) need to be modified. The appropriate left scattering data consists of R for $k \in \mathbf{R} \setminus [-c, c]$, $|T|$ for $k \in [-c, c]$, κ_j and $c_{l;j}$ for $j = 1, \dots, N$. One has

$$g_l(\alpha) := -\hat{R}(\alpha) - \frac{1}{2\pi} \int_0^c dk\, |T(k)|^2 \, e^{-\sqrt{c^2-k^2}\,\alpha} - \sum_{j=1}^N c_{l;j}^2 \, e^{-\sqrt{\kappa_j^2+c^2}\,\alpha}.$$

where

$$\hat{R}(\alpha) := \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi\, R(k(\xi))\, e^{i\xi\alpha}, \tag{5.2}$$

with $k(\xi)$ denoting the map $\xi \mapsto k$. The potential is obtained from the solution $B_l(x, \alpha)$ of the Marchenko equation (2.4) as

$$V(x) = c^2 - 2\, \frac{dB_l(x, 0^+)}{dx}. \tag{5.3}$$

The Jost solution from the left is constructed as

$$f_l(k, x) = e^{i\sqrt{k^2-c^2}x} \left[1 + \int_0^\infty d\alpha\, B_l(x, \alpha)\, e^{i\sqrt{k^2-c^2}\alpha} \right].$$

One can give a characterization of the scattering data $\{T_l, T_r, R, L, \{\kappa_j\}, \{c_{l;j}\}, \{c_{r;j}\}\}$ so that it corresponds to a unique potential V that is real valued and satisfies $V \in L_1^1(\mathbf{R}^-)$ and $V - c^2 \in L_1^1(\mathbf{R}^+)$. When the following conditions are satisfied, both the right and left Marchenko equations are uniquely solvable, and the right hand sides of (2.7) and (5.3) are equal to each other, and V is obtained from (2.7) or (5.3).

(i) The right transmission coefficient T_r is related to the left transmission coefficient $T := T_l$ as in (5.1), and the right reflection coefficient is related to the left scattering coefficients as in (1.30). The scattering coefficients T, R, and L satisfy (1.10) and (1.11).

(ii) The reflection coefficients L and R are continuous on $\mathbf{R}$ and belong to $L^2(\mathbf{R})$.

(iii) The left transmission coefficient T is meromorphic in $\mathbf{C}^+$ with simple poles at $k = i\kappa_j$, it is continuous in $\overline{\mathbf{C}^+} \setminus \{i\kappa_1, \dots, i\kappa_N\}$, and the residues of $T(k)$ at $k = i\kappa_j$ satisfy (1.19).

(iv) The scattering coefficients satisfy

$$1 - |R(k)|^2 = 1 - |L(k)|^2 = \frac{\xi}{k}\,|T(k)|^2 \neq 0, \qquad k \in \mathbf{R} \setminus [-c, c], \tag{5.4}$$

$$R(k) = -1, \quad L(k) = \frac{T(k)}{T^*(k)}, \qquad k \in [-c, c]. \tag{5.5}$$

(v) The reflection coefficients L and R are $o(1/k)$ as $k \to \pm\infty$. The left transmission coefficient T is nonzero in $\overline{\mathbf{C}^+} \setminus \{0, i\kappa_1, \ldots, i\kappa_N\}$, and $T(k) = 1 + O(1/k)$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$.

(vi) The scattering coefficients T, R, and L are continuous at $k = 0$. As $k \to 0$ in $\overline{\mathbf{C}^+}$ one has (Akt 99)

$$T(k) = \begin{cases} \dfrac{2ik}{W_0} + o(k), & \text{generic case,} \\[2ex] \dfrac{2}{\gamma_0} + o(1), & \text{exceptional case,} \end{cases}$$

where W_0 and γ_0 are some real nonzero constants (they are actually the constants defined in (1.20) and (1.21), respectively), and as $k \to 0$ in $\mathbf{R}$

$$R(k) = -1 + o(1), \qquad L(k) = \begin{cases} -1 + o(1), & \text{generic case,} \\ 1 + o(1), & \text{exceptional case.} \end{cases}$$

(vii) $\hat{L}$ defined in (2.11) and $\hat{R}$ defined in (5.2) are absolutely continuous and belong to $L^2(\mathbf{R})$; moreover, $\hat{L}' \in L_1^1(-\infty, a)$ and $\hat{R}' \in L_1^1(a, +\infty)$ for any fixed $a \in \mathbf{R}$.

2.4.6. PHASE RECOVERY

In X-ray and neutron reflectometry, one needs to determine the scattering length density of an unknown layered material, and this is done by measuring the intensity of a probing beam reflected off that material as a function of the angle of incidence. Mathematically, this amounts to determining a portion of a steplike potential in the 1-D Schrödinger equation when one knows the rest of the potential and some reflectivities (i.e. amplitudes of some reflection coefficients without their phases).

Consider (1.1) with a real-valued potential V that can be written as $V = V_1 + V_2$, where V_1 is supported in $\mathbf{R}^-$, V_2 is supported in $\mathbf{R}^+$, $V_1 \in L_1^1(\mathbf{R}^-)$, and $V_2 - c^2 \in L_1^1(\mathbf{R}^+)$ for some $c \geq 0$. Let R_1 and T_1 denote the reflection coefficient from the right and the transmission coefficient for V_1, L_2 and T_2 the reflection and transmission coefficients from the left for V_2, and L and T the reflection and transmission coefficients from the left for V. For simplicity, we assume that neither V_1 nor V_2 have any bound states and refer the reader to (Akt 00b) for the phase recovery problem with bound states.

Mathematically, one needs to determine V_2 from the scattering data $\{R_1, |L_2|, |L|\}$. In other words, we are missing the phases of L_2 and L and only know their amplitudes. Because of (5.5), we have $|L_2(k)| = 1$ and $|L(k)| = 1$ for $k \in [-c, c]$. Thus, the data for $k \in [-c, c]$ is useful only to recover the value of c and otherwise does not directly contribute to the phase recovery. As seen from Section 2.4.5, V_2 is recovered when L_2 is obtained for $k \in \mathbf{R}$ from the given data. This inverse problem can be solved as follows. From a generalization of (1.40) to steplike potentials, we have

$$\frac{1}{T(k)} = \frac{1 - R_1(k)\, L_2(k)}{T_1(k)\, T_2(k)}, \qquad k \in \mathbf{R} \setminus \{0\}. \tag{6.1}$$

Define

$$F(k) := 1 - R_1(k)\, L_2(k), \qquad k \in \mathbf{R}. \tag{6.2}$$

From (6.1) we see that

$$F(k) = \frac{T_1(k)\, T_2(k)}{T(k)}. \tag{6.3}$$

Using (5.4) and (6.3) we get

$$|F|^2 = \frac{(1 - |R_1|^2)(1 - |L_2|^2)}{1 - |L|^2}, \qquad k \in \mathbf{R} \setminus [-c, c]. \tag{6.4}$$

On the other hand, from (6.2) it follows that

$$|F|^2 = -1 + |R_1|^2 |L_2|^2 + 2 \operatorname{Re} F, \qquad k \in \mathbf{R},$$

which, combined with (6.4), yields the real part of F as

$$\operatorname{Re} F = \frac{1}{2} \left[1 - |R_1|^2 |L_2|^2 + \frac{(1 - |R_1|^2)(1 - |L_2|^2)}{1 - |L|^2} \right], \qquad k \in \mathbf{R} \setminus [-c, c]. \tag{6.5}$$

As seen from (6.4) and (6.5), our data uniquely determines both $|F|$ and $\operatorname{Re} F$ at each $k \in \mathbf{R} \setminus [-c, c]$. Moreover, because of (6.3), $F(k)$ has an analytic extension from $\mathbf{R}$ to $\mathbf{C}^+$ which is continuous in $\overline{\mathbf{C}^+}$ and free of zeros in $\overline{\mathbf{C}^+} \setminus \{0\}$. Thus, one can determine F uniquely for $k \in \overline{\mathbf{C}^+}$. Then L_2 is recovered as $L_2 = (1 - F)/R_1$ for all $k \in \mathbf{R}$, from which V_2 is uniquely constructed. Under certain restrictions, an exact quadrature is known (Akt 00b) for the analytic continuation of F or L_2 from $k \in \mathbf{R} \setminus [-c, c]$ to $k \in [-c, c]$, and that quadrature is also easy to implement numerically.

Assume that V_1 is replaced by another potential $\tilde{V}_1$ whose reflection coefficient from the right is $\tilde{R}_1$. Let $\tilde{V} := \tilde{V}_1 + V_2$, and use $\tilde{L}$ to denote the reflection coefficient from the left for $\tilde{V}$. If we use $\{R_1, \tilde{R}_1, |L_2|, |L|, |\tilde{L}|\}$ as our scattering data, then

$$L_2 = \frac{R_1 - \tilde{R}_1 + |L_2|^2 \, R_1 \, \tilde{R}_1 \, [\tilde{R}_1^* - R_1^*] + [1 - |L_2|^2] \, \Upsilon}{2i \operatorname{Im} \{R_1 \, \tilde{R}_1^*\}}, \qquad k \in \mathbf{R} \setminus [-c, c],$$

where we have defined

$$\Upsilon := \frac{\tilde{R}_1 \, (1 - |R_1|^2)}{1 - |L|^2} - \frac{R_1 \, (1 - |\tilde{R}_1|^2)}{1 - |\tilde{L}|^2}.$$

Thus, the new data determines L_2 provided $\operatorname{Im} \{R_1 \tilde{R}_1^*\} \neq 0$ in some interval in $\mathbf{R} \setminus [-c, c]$. This forms the basis for the *three-measurement method* (Maj 95, de Haa 95), so called because the data contains three sets of reflectivities compared to the earlier data $\{R_1, |L_2|, |L|\}$ containing only two sets of reflectivities. It is possible to have $\operatorname{Im} \{R_1 \tilde{R}_1^*\} = 0$ for all $k \in \mathbf{R}$ for two distinct potentials V_1 and $\tilde{V}_1$; if this happens $\{R_1, \tilde{R}_1, |L_2|, |L|, |\tilde{L}|\}$ and $\{R_1, |L_2|, |L|\}$ contain exactly the same information.

In the special case when V_2 is constant, one has $L_2(k) = (k - \xi)/(k + \xi)$, where ξ is the quantity appearing in (5.1). Then, via (6.2), one has

$$\operatorname{Re} \{R_1(k)\} = -\frac{1}{L_2(k)} \operatorname{Re} \{F(k)\}, \qquad k \in \mathbf{R} \setminus [-c, c],$$

and hence, because of (6.5), knowledge of $\{|R_1|, L_2, |L|\}$ at one k value with $k > c$ uniquely and explicitly determines $\operatorname{Re}\{R_1(k)\}$ at that k value. Moreover, R_1 has a unique analytic continuation from any interval on $\mathbf{R}$ to $\overline{\mathbf{C}^+}$. Thus, knowledge of $\operatorname{Re} \{R_1(k)\}$ on any such interval is sufficient to construct V_1. This is the basic idea behind the method of *variation of surrounding media* (Maj 98), which is an experimentally feasible procedure in neutron reflectometry to determine the scattering length density of an unknown layered medium. The reader is referred to (de Haa 95, Maj 95, Maj 98, Akt 00b) and the references therein for further studies on various physical and mathematical aspects of neutron reflectometry.

2.4.7. INVERSE PROBLEMS FOR NONHOMOGENEOUS MEDIA

Consider the generalized Schrödinger equation

$$\frac{d^2\psi(k,x)}{dx^2}+k^2\,H(x)^2\,\psi(k,x)=Q(x)\,\psi(k,x),\qquad x\in\mathbf{R},\tag{7.1}$$

where Q is real valued and belongs to $L^1_1(\mathbf{R})$, H is real valued, strictly positive, and $H-1\in L^1(\mathbf{R})$. Switching to the *travel-time coordinate*

$$y=y(x):=\int_0^x dt\,H(t),\tag{7.2}$$

via the *Liouville transformation*

$$\phi(k,y(x)):=\sqrt{H(x)}\,\psi(k,x),\tag{7.3}$$

one can transform (7.1) into an equation of the form (1.1) that is given by

$$\frac{d^2\phi(k,y)}{dy^2}+k^2\,\phi(k,y)=V(y)\,\phi(k,y),\qquad y\in\mathbf{R},\tag{7.4}$$

with

$$V(y(x)):=\frac{Q(x)}{H(x)^2}+\frac{H''(x)}{2H(x)^3}-\frac{3\,H'(x)^2}{4H(x)^4}.\tag{7.5}$$

When $V\in L^1_1(\mathbf{R})$, one can recover $Q(x)$ from any scattering data that implies knowledge of $V(y)$ and the relation between x and y.

Let τ, ℓ, and ρ denote the transmission coefficient, the reflection coefficient from the left, and the reflection coefficient from the right, respectively, for (7.4), and let T, L, and R denote the corresponding coefficients for (7.1). These two sets are related to each other by

$$\tau(k)=T(k)\,e^{ikA},\quad \ell(k)=L(k)\,e^{2ikA_-},\quad \rho(k)=R(k)\,e^{2ikA_+},$$

where

$$A_\pm := \pm \int_0^{\pm\infty} dt\,[1 - H(t)], \quad A := A_- + A_+.$$

Let $f_l^{[0]}(0,x)$ be the zero-energy Jost solution from the left corresponding to Q. Note that Q uniquely determines $f_l^{[0]}(0,x)$ and vice versa. For example, given Q we can determine $f_l^{[0]}(0,x)$ by solving

$$f_l^{[0]}(0,x) = 1 + \int_x^\infty dt\,(t-x)\,Q(t)\,f_l^{[0]}(0,t),$$

and given $f_l^{[0]}(0,x)$ one can determine Q by using

$$Q(x) = \frac{1}{f_l^{[0]}(0,x)}\frac{d^2 f_l^{[0]}(0,x)}{dx^2}. \tag{7.6}$$

Note also that

$$f_l^{[0]}(0,x) = f_l(0,x), \qquad x \in \mathbf{R}, \tag{7.7}$$

where $f_l(k,x)$ is the Jost solution from the left for (7.1).

Let $Z_l(k,y)$ denote the Faddeev function from the left for (7.4), i.e. $e^{iky}Z_l(k,y)$ is the Jost solution from the left for (7.4). Assuming that our scattering data consists of Q, ρ, and the bound-state energies and norming constants for (7.4), we can recover $V(y)$ by solving the Marchenko equation using this data as the input. Having found $V(y)$, we also have $Z_l(0,y)$ at hand because $V(y) = Z_l''(0,y)/Z_l(0,y)$ and

$$Z_l(0,y) = 1 + o(1), \quad Z_l'(0,y) = o(1), \qquad y \to +\infty.$$

The relationship between x and y can be obtained by solving the algebraic equation

$$\int_0^y \frac{ds}{Z_l(0,s)^2} = \int_0^x \frac{dt}{f_l^{[0]}(0,t)^2},$$

which is obtained with the help of (7.3) at $k = 0$. Once we have $y = y(x)$, then H is obtained as $H(x) = dy(x)/dx$.

On the other hand, suppose we are given H and are interested in the recovery of Q. Using (7.2) one obtains y as a function of x. Then, from the data consisting of ρ and the bound-state energies and norming constants, one obtains $V(y)$. Having $V(y)$ and $y = y(x)$ at hand, one uses (7.5) to recover $Q(x)$.

The reader is referred to (Akt 92a, Akt 92b) and the references therein for the details and variations of the methods outlined above. Such methods also have generalizations when H and H' have jump discontinuities and H has distinct asymptotics as $x \to \pm\infty$, which is discussed in Section 2.4.8.

2.4.8. NONHOMOGENEOUS MEDIA WITH JUMP DISCONTINUITIES

Consider again the generalized Schrödinger equation (7.1) describing wave propagation in a nonhomogeneous, elastic medium. In order to reflect abrupt changes in the material properties of the medium, H is now allowed to have jump discontinuities at a finite number of points. We also let $H(x)$ to have different asymptotics as $x \to \pm\infty$. When Q is known, we are interested in reconstructing H from an appropriate set of scattering data, which will be specified.

The assumptions on H and Q are as follows:

(i) H is strictly positive and piecewise continuous with jump discontinuities at x_j for $j = 1, \ldots, n$ such that $x_1 < \cdots < x_n$.

(ii) $H(x) \to H_\pm$ as $x \to \pm\infty$, where $H_\pm$ are positive constants.

(iii) $H - H_\pm \in L^1(\mathbf{R}^\pm)$.

(iv) H' is absolutely continuous on (x_j, x_{j+1}) and $2H''H - 3(H')^2 \in L^1_1(x_j, x_{j+1})$ for $j = 0, \ldots, n$, where $x_0 := -\infty$ and $x_{n+1} := +\infty$.

(v) $Q \in L^1_2(\mathbf{R})$.

(vi) There are no bound states for (7.1).

One can weaken (v) to just $Q \in L^1_1(\mathbf{R})$, which is sufficient (Akt 96a) for the reconstruction of H. However, not all technical questions regarding the uniqueness of the inversion procedure (Akt 95, Akt 96a) have been answered when Q is only in $L^1_1(\mathbf{R})$. The inversion when there are bound states can be found in (Akt 95). There are no bound states if, for example, $Q(x) \geq 0$. Because of (iv) a Liouville transformation can be used in each subinterval (x_j, x_{j+1}).

Various authors have studied inverse scattering problems for differential equations with discontinuous coefficients; for example, see (War 69, Kru 82, Sab 88, Gri 91a, Gri 91b). The work most directly related to the method presented here is that of Grinberg, who, in the special case $Q \equiv 0$, analyzed the recovery of H using the solution of a singular integral equation. When $Q \not\equiv 0$ the analysis of the problem becomes more involved and there are essential differences in the results as compared to the case $Q \equiv 0$.

There are some difficulties and new twists in this inverse problem that one does not encounter in the inverse problem for (1.1). There are also other interesting issues; for example, a question of practical importance is whether certain characteristic properties of H can be recovered more quickly, that is, without having to solve the inverse problem first. Quantities that fall into this category include the number of discontinuities, the ratios $H(x_j^-)/H(x_j^+)$, and the integrals $\int_0^{x_j} dz\, H(z)$ representing the time it takes for the wave to travel from the origin to the discontinuity x_j. It turns out that such information can be extracted from the large-k asymptotics of the reflection and transmission coefficients (Gri 91a, Akt 96b).

The Jost solutions f_l and f_r of (7.1) satisfy

$$f_l(k,x) = \begin{cases} e^{ikH_+x} + o(1), & x \to +\infty, \\[2mm] \dfrac{1}{T_l(k)}\, e^{ikH_-x} + \dfrac{L(k)}{T_l(k)}\, e^{-ikH_-x} + o(1), & x \to -\infty, \end{cases}$$

$$f_r(k,x) = \begin{cases} \dfrac{1}{T_r(k)}\, e^{-ikH_+x} + \dfrac{R(k)}{T_r(k)}\, e^{ikH_+x} + o(1), & x \to +\infty, \\[2mm] e^{-ikH_-x} + o(1), & x \to -\infty, \end{cases}$$

where T_l and T_r are the transmission coefficients from the left and right, respectively, and L and R are the reflection coefficients from the left and right, respectively. Note that $T_l(k)$ and $T_r(k)$ are different unless $H_+ = H_-$, and they are related by $H_+\, T_l(k) = H_-\, T_r(k)$.

As for (1.1) we need to distinguish between the generic case and the exceptional case; the definitions and notation are the same as in Section 2.4.1, and the division into a generic case and an exceptional case is solely governed by Q because H does not affect the solutions of (7.1) at $k = 0$. In particular, if $Q \equiv 0$ then the exceptional case occurs, and if $Q(x) \geq 0$ but $Q \not\equiv 0$ then the generic case occurs.

Using the travel-time coordinate (7.2), one can still transform (7.1) into (7.4); even though the potential $V(y)$ given in (7.5) cannot be defined at $y_j := y(x_j)$, we have $V \in L_1^1(y_j, y_{j+1})$ for $j = 0, \ldots, n$. The function $\phi(k, y)$ defined in (7.3) and $\phi'(k, y)$ are not continuous at y_j, but they satisfy certain matching conditions (Akt 95) involving the jumps in H and in H'/H.

Define the Faddeev functions $Z_l(k, y)$ and $Z_r(k, y)$ associated with (7.1) as

$$Z_l(k,y) := \sqrt{\frac{H(x)}{H_+}}\, e^{-iky-ikA_+} f_l(k,x), \quad Z_r(k,y) := \sqrt{\frac{H(x)}{H_-}}\, e^{iky-ikA_-} f_r(k,x),$$

where

$$A_\pm := \pm \int_0^{\pm\infty} dz\, [H_\pm - H(z)], \quad A := A_- + A_+.$$

For each $y \in \mathbf{R} \setminus \{y_1, \ldots, y_n\}$, the Faddeev functions $Z_l(\cdot, y)$ and $Z_r(\cdot, y)$ are continuous on $\overline{\mathbf{C}^+}$, analytic on $\mathbf{C}^+$, behave like $1 + O(1/k)$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$, and satisfy the Riemann-Hilbert problem

$$\begin{bmatrix} Z_l(-k,y) \\ Z_r(-k,y) \end{bmatrix} = \begin{bmatrix} \tau(k) & -\rho(k)\, e^{2iky} \\ -\ell(k)\, e^{-2iky} & \tau(k) \end{bmatrix} \begin{bmatrix} Z_r(k,y) \\ Z_l(k,y) \end{bmatrix}, \qquad k \in \mathbf{R}, \tag{8.1}$$

where the *reduced scattering coefficients* are given by

$$\tau(k) := \sqrt{\frac{H_+}{H_-}}\, T_l(k)\, e^{ikA}, \quad \ell(k) := L(k)\, e^{2ikA_-}, \quad \rho(k) := R(k)\, e^{2ikA_+}. \tag{8.2}$$

Define

$$X(k,y) := \frac{i\sqrt{H_+}}{k\,\sqrt{H(x)}\, f_l(0,x)}\, [Z_l(-k,y) - Z_l(0,y)]\,. \tag{8.3}$$

The notation is justified, since it turns out that the right hand side of (8.3) does not depend explicitly on x, so the left hand side depends on x only through y. The Riemann-Hilbert problem (8.1) can be transformed into the singular integral equation

$$X(k,y) = X_0(k,y) + (\mathcal{O}_y X)(k,y), \tag{8.4}$$

where

$$X_0(k,y) := \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{ds}{s-k+i0^+}\,\frac{\rho(s)\,e^{2isy}-\rho(0)}{s}, \tag{8.5}$$

$$(\mathcal{O}_y X)(k,y) := \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{ds}{s+k-i0^+}\rho(-s)\,e^{-2isy}\,X(s,y). \tag{8.6}$$

The integral equation (8.4) is uniquely solvable in the Hardy spaces $\mathbf{H}^p_-(\mathbf{R})$ with $1<p<+\infty$ when $Q\in L^1_2(\mathbf{R})$. Under the weaker assumption $Q\in L^1_{1+\alpha}(\mathbf{R})$, the unique solvability holds for $p<(1-\alpha)^{-1}$.

Note that (8.4) uses $\rho(k)$ as the input. In order to use $R(k)$ as the input in (8.4), we can exploit (8.2), where $\rho(k)$ and $R(k)$ are related by the phase factor involving A_+, which can be handled by a shift in y in the solution of (8.4). If $\widetilde{X}(k,y)$ denotes the solution of (8.4) obtained by replacing $\rho(k)$ by $R(k)$ in (8.5) and (8.6), then we get

$$\widetilde{X}(k,y-A_+) = X(k,y). \tag{8.7}$$

Using (8.1), (8.3), and (8.7) one obtains

$$y + A_+ + \widetilde{X}(0,y+A_+) = H_+\,G_1(x), \tag{8.8}$$

where (Akt 96b)

$$G_1(x) := -\int_0^{\infty} dz\,\frac{1-f_l(0,z)^2}{f_l(0,z)^2} + \int_0^x \frac{dz}{f_l(0,z)^2}.$$

When $Q\in L^1_2(\mathbf{R})$, one has $1-f_l(0,\cdot)^2\in L^1(\mathbf{R}^+)$; because of (7.6) and (7.7), $G_1(x)$ is determined by $Q(x)$. We remark that (8.8) is an implicit equation for $y(x)$. The constant A_+ is determined by using the condition $y(0)=0$ in (7.2), which results in

$$A_+ + \widetilde{X}(0,A_+) = H_+\,G_1(0). \tag{8.9}$$

It can be shown that this equation determines A_+ uniquely (Akt 96b).

The recovery of H is accomplished as follows. First, one solves (8.4) with $\rho(k)$ replaced by $R(k)$ and obtains $\widetilde{X}(k, y)$. Next, (8.9) is solved for A_+ and then $y(x)$ is obtained by solving (8.8). As a final step, one obtains $H(x) = y'(x)$ by differentiating $y(x)$.

One can analyze (8.8) in more detail (Akt 96b). In the exceptional case, the constant H_+ is a free parameter and has to be specified for a unique recovery of H. In the generic case, the situation is a bit different. The limits $w_0 := \lim_{z\to-\infty}[z+\widetilde{X}(0, z)]$ and $G_1(-\infty) := \lim_{x\to-\infty} G_1(x)$ exist and are finite, so that, by (8.8), we have $H_+ = w_0/G_1(-\infty)$ provided $G_1(-\infty) \neq 0$; in other words, H_+ is not a free parameter, and it is fixed by the data $\{R, Q\}$. However, if $G_1(-\infty) = 0$, we must also have $w_0 = 0$ in order for (8.8) to be solvable for y as a function of x. One can show that if $G_1(-\infty) = 0$, then H_+ is also a free parameter as in the exceptional case. From these facts it follows that the proper scattering data to recover H is given by $\{R, H_+, Q\}$ in the exceptional case, by $\{R, H_+, Q\}$ in the generic case with $G_1(-\infty) = 0$, and by $\{R, Q\}$ in the generic case with $G_1(-\infty) \neq 0$.

In summary, under the assumptions (i)-(vi), the solution of the inverse scattering problem is unique when the scattering data is chosen as indicated above. Moreover, in the generic case with $G_1(-\infty) = 0$ and in the exceptional case, the constant H_+ is a free parameter in the sense that for any choice of $H_+ > 0$, the function H resulting from the solution of (8.8) corresponds to the same reflection coefficient R.

For the recovery of H with jump discontinuities by using a Marchenko method, the reader is referred to (Akt 96a).

2.4.9. OTHER EQUATIONS

Consider the *impedance-potential equation* (Sab 88, Cha 89)

$$\frac{1}{p(x)^2}\frac{d}{dx}\left[p(x)^2\,\frac{d\psi(k,x)}{dx}\right] + k^2\,\psi(k,x) = Q(x)\,\psi(k,x), \qquad x \in \mathbf{R}, \tag{9.1}$$

where Q is real valued, $Q \in L^1_1(\mathbf{R})$, p is strictly positive, $p'' \in L^1_1(\mathbf{R})$, and $p - 1 \in L^1(\mathbf{R})$. Letting $\phi(k, x) := p(x)\,\psi(k, x)$ one can transform (9.1) into an equation of the form (1.1) that is given by

$$\frac{d^2\phi(k,x)}{dx^2}+k^2\,\phi(k,x)=\left[Q(x)+\frac{p''(x)}{p(x)}\right]\phi(k,x),\qquad x\in\mathbf{R}.$$

Alternatively, using

$$y=y(x):=\int_0^x \frac{dt}{p(t)^2},\quad \varphi(k,y(x)):=\psi(k,x),$$

one can transform (9.1) into an equation similar to (7.1) that is given by

$$\frac{d^2\varphi(k,y)}{dy^2}+k^2\,n(y)^2\,\varphi(k,y)=W(y)\,\varphi(k,y),\qquad y\in\mathbf{R},$$

where

$$n(y(x)):=p(x)^2,\quad W(y(x)):=Q(x)\,p(x)^4.$$

The reader is referred to (Cha 89) for the analysis of (9.1) when $p(x)$ and $p'(x)$ has jump discontinuities.

For the inverse scattering problems related to the matrix Schrödinger equation, the reader is referred to (Olm 85, Cha 89) and the references therein. Under certain restrictions, the $n\times n$ matrix potential is recovered via the Marchenko method from an appropriate set of scattering data containing an $n\times n$ matrix reflection coefficient and the bound-state information.

Consider the inverse scattering problem of recovery of $P(x)$ and $Q(x)$ in

$$\psi''(k,x)+k^2\,\psi(k,x)=[ik\,P(x)+Q(x)]\,\psi(k,x),\qquad x\in\mathbf{R}. \tag{9.2}$$

When $P(x)$ is purely imaginary and $Q(x)$ is real the reader is referred to (Jau 76b, Jau 76c, Sat 95). A Marchenko inversion method can be formulated utilizing the scattering data from (9.2) as well as the data from the equation obtained when $P(x)$ is changed to $-P(x)$ in (9.2). If P is real valued, then (9.2) is nonselfadjoint, and the inverse problem (Jau 76a, Akt 98a) becomes challenging due the nonunitarity of the scattering matrix, possible singularities of the transmission coefficient at real k values, and some complications related to bound states.

For the inverse problems related to first-order systems containing a matrix potential, the reader is referred to (Bea 87, Cha 89) as starting references for further reading. Such inverse problems are usually formulated as a matrix Riemann-Hilbert problem, from whose solution the matrix potential is constructed.

The connection between the inverse scattering for (1.1) and the initial-value problem for the Korteweg-de Vries equation (KdV) is treated in Section 6.2.1. Here we briefly mention the recent work by Sabatier (Sab 99) on the recovery of the solution $u(x,t)$ of the KdV equation for all $x,t \in \mathbf{R}$ when $u(0,t)$, $u_x(0,t)$, and $u_{xx}(0,t)$ are known for all $t \in \mathbf{R}$, and the study on "elbow scattering" (Sab 00a) related to the KdV and linearized KdV equations, where the scattering and inverse scattering problems are analyzed on two perpendicular half lines meeting at a point on the xt-plane.

For various other inverse problems on the line, we refer the reader to Chapter 17 of (Cha 89), the recent review paper by Sabatier (Sab 00b), and the references therein.

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